2 Infinite Games

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2.1 Introduction

This chapter is meant as an introduction to infinite two-person games on directed graphs. We will define what they are, how they are played, what exactly a strategy is, what we mean when we say a game is won by a certain player, etc. We will introduce fundamental notions such as determinacy, forgetful strategies, memoryless strategies, and so on. And we will state fundamental results, which will be proved in later chapters.

2.2 Games

A game is composed of an arena and a winning condition. We will first study arenas and then add winning conditions on top of arenas.

2.2.1 Arenas

An **arena** is a triple

$$\mathcal{A} = (V_0, V_1, E) \tag{2.1}$$

where V_0 is a set of **0-vertices**, V_1 a set of **1-vertices**, disjoint from V_0, and $E \subseteq (V_0 \cup V_1) \times (V_0 \cup V_1)$ is the **edge relation,** sometimes also called the set of moves. The union of V_0 and V_1 is denoted V. Observe that with this notation the requirement for the edge relation reads $E \subseteq V \times V$. The set of **successors** of $v \in V$ is defined by $vE = \{ v' \in V \mid (v, v') \in E \}$.

The games we are interested in are played by two players, called **Player 0** and **Player 1**. We will often fix $\sigma \in \{0, 1\}$ and consider Player σ; if we then want to refer to the other player, we will speak of him or her as Player σ's opponent and write Player $\overline{\sigma}$. Formally, we set $\overline{\sigma} = 1 - \sigma$ for $\sigma \in \{0, 1\}$.

Observe that there is no restriction on the number of the successors of a vertex in an arena. Also, we don't require that (V, E) is a bipartite graph with corresponding partition $\{V_0, V_1\}$.

2.2.2 Plays

A play of a game with an arena as above may be imagined in the following way: a token is placed on some initial vertex $v \in V$. If v is a 0-vertex then Player 0

E. Grädel et al. (Eds.): Automata, Logics, and Infinite Games, LNCS 2500, pp. 23-38, 2002.

moves the token from v to a successor $v' \in vE$ of v; symmetrically, if v is a 1-vertex then Player 1 moves the token from v to a successor $v' \in vE$ of v. More concisely, when v is a σ-vertex, then Player σ moves the token from v to $v' \in vE$. Next, when v' is a σ-vertex, then Player σ moves the token from v' to $v'' \in v'E$. This is repeated either infinitely often or until a vertex $\overline{v}$ without successors, a **dead end**, is reached. Formally, a vertex $\overline{v}$ is called a dead end if $\overline{v}E = \emptyset$.

We define a **play** in the arena $\mathcal{A}$ as above as being either

- an infinite path $\pi = v_0 v_1 v_2 \cdots \in V^\omega$ with $v_{i+1} \in v_i E$ for all $i \in \omega$ (**infinite play**) or
- a finite path $\pi = v_0 v_1 \cdots v_l \in V^+$ with $v_{i+1} \in v_i E$ for all $i < l$, but $v_l E = \emptyset$ (**finite play**).

A **prefix of a play** is defined in the obvious way.

Now that we know what arenas and plays are we need to explain what kind of winning conditions we are going to use and how arenas together with winning conditions make games.

2.2.3 Games and Winning Sets

Let $\mathcal{A}$ be an arena as above and $\mathrm{Win} \subseteq V^\omega$. The pair

$$(\mathcal{A}, \mathrm{Win}) \tag{2.2}$$

is then called a **game**, where $\mathcal{A}$ is the arena of the game and Win its **winning set**. The plays of that game are the plays in the arena $\mathcal{A}$. Player 0 is declared the **winner of a play** π in the game $\mathcal{G}$ iff

- π is a finite play $\pi = v_0 v_1 \cdots v_l \in V^+$ and v_l is a 1-vertex where Player 1 can't move anymore (when v_l is a dead end) or
- π is an infinite play and $\pi \in \mathrm{Win}$.

Player 1 wins π if Player 0 does not win π.

2.2.4 Winning Conditions

We will only be interested in winning sets that can be described using the acceptance conditions that were discussed in the previous chapter. But recall that these acceptance conditions made only sense when used with automata with a finite state space—a run of an infinite-state automaton might have no recurring state. We will therefore colour the vertices of an arena and apply the acceptance conditions from the previous chapter on colour sequences.

Let $\mathcal{A}$ be as above and assume $\chi\colon V \to C$ is some function mapping the vertices of the arena to a finite set C of so-called colours; such a function will be called a **colouring function**. The colouring function is extended to plays in a straightforward way. When $\pi = v_0 v_1 \cdots$ is a play, then its colouring, $\chi(\pi)$, is given by $\chi(\pi) = \chi(v_0)\chi(v_1)\chi(v_2)\cdots$. So, when C is viewed as the state set of a finite ω-automaton and Acc is an acceptance condition for this automaton (in the sense of the previous chapter), then we will write $W_\chi(\text{Acc})$ for the winning set consisting of all infinite plays π where $\chi(\pi)$ is accepted according to Acc. Depending on the actual acceptance condition we are interested in, this means the following, where π stands for any element of V^ω.

- Muller condition ($\text{Acc} = \mathcal{F} \subseteq \mathscr{P}_0(C)$): $\pi \in W_\chi(\text{Acc})$ iff $\text{Inf}(\chi(\pi)) \in \mathcal{F}$.
- Rabin condition ($\text{Acc} = \{(E_0, F_0), (E_1, F_1), \ldots, (E_{m-1}, F_{m-1})\}$): $\pi \in W_\chi(\text{Acc})$ iff $\exists k \in [m]$ such that $\text{Inf}(\chi(\pi)) \cap E_k = \emptyset$ and $\text{Inf}(\chi(\pi)) \cap F_k \neq \emptyset$,
- Streett condition ($\text{Acc} = \{(E_0, F_0), (E_1, F_1), \ldots, (E_{m-1}, F_{m-1})\}$): $\pi \in W_\chi(\text{Acc})$ iff $\forall k \in [m].(\text{Inf}(\chi(\pi)) \cap E_k \neq \emptyset \ \vee \ \text{Inf}(\chi(\pi)) \cap F_k = \emptyset)$,
- Rabin chain condition ($\text{Acc} = \{(E_0, F_0), (E_1, F_1), \ldots, (E_{m-1}, F_{m-1})\}$ where $E_0 \subset F_0 \subset E_1 \subset F_1 \subset \ldots \subset E_{m-1} \subset F_{m-1}$): like the Rabin condition.
- Parity conditions (the colour set C is a finite subset of the integers):
  - max-parity condition: $\pi \in W_\chi(\text{Acc})$ iff $\max(\text{Inf}(\chi(\pi)))$ is even.
  - min-parity condition: $\pi \in W_\chi(\text{Acc})$ iff $\min(\text{Inf}(\chi(\pi)))$ is even.
- Büchi condition ($\text{Acc} = F \subseteq C$): $\pi \in W_\chi(\text{Acc})$ iff $\text{Inf}(\chi(\pi)) \cap F \neq \emptyset$.
- 1-winning ($\text{Acc} = F \subseteq C$): $\pi \in W_\chi(\text{Acc})$ iff $\text{Occ}(\chi(\pi)) \cap F \neq \emptyset$.

For simplicity, we will just write $(\mathcal{A}, \chi, \text{Acc})$ instead of $(\mathcal{A}, W_\chi(\text{Acc}))$. To indicate that we are working with a certain acceptance/winning condition, we will speak of **Muller, Büchi, ... games**. We will say a game is a **regular game** if its winning set is equal to $W_\chi(\text{Acc})$ for some χ and some acceptance condition Acc from above, except for 1-acceptance.

Example 2.1. Let $\mathcal{A} = (V_0, V_1, E, \chi)$ be the (coloured) arena presented in Figure 2.1. We have the 0-vertices $V_0 = \{z_1, z_2, z_5, z_6\}$ (circles) and the 1-vertices $V_1 = \{z_0, z_3, z_4\}$ (squares). The colours are $C = \{1, 2, 3, 4\}$. The edge relation E and the colour mapping χ may be derived from the picture, i.e. $\chi(z_4) = 2$ or $\chi(z_0) = 1$. Note that we don't have a dead end in our example. As a winning condition we choose the Muller acceptance condition given by $\mathcal{F} = \{\{1, 2\}, \{1, 2, 3, 4\}\}$.

A possible infinite play in this game is $\pi = z_6 z_3 z_2 z_4 z_2 z_4 z_6 z_5 (z_2 z_4)^\omega$. This play is winning for Player 0 because $\chi(\pi) = 23121224(12)^\omega$ and $\text{Inf}(\chi(\pi)) = \{1, 2\} \in \mathcal{F}$. The play $\pi' = (z_2 z_4 z_6 z_3)^\omega$ yields $\chi(\pi') = (1223)^\omega$ and $\text{Inf}(\chi(\pi')) = \{1, 2, 3\} \notin \mathcal{F}$. Hence π' is winning for Player 1.

When we want to fix a vertex where all plays we consider should start, we add this vertex to the game: an **initialized game** is a tuple $(\mathcal{G}, v_I)$ where v_I is a vertex of the arena of $\mathcal{G}$. A play of such a game is a play of the uninitialized game which starts in v_I.

2.3 Fundamental Questions

There are several obvious questions to ask when one is confronted with an initialized game as introduced in the previous section.

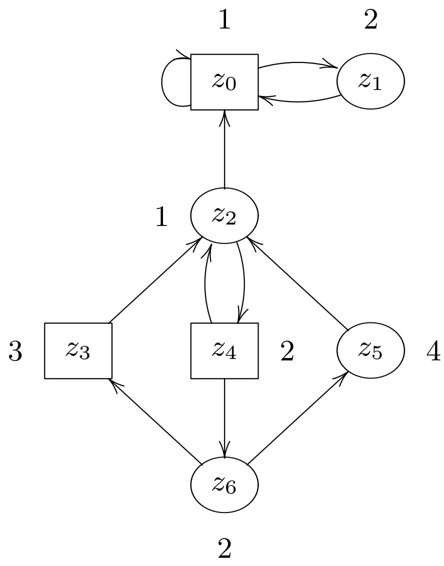

Fig. 2.1. Coloured example arena

First, it would be interesting to know if one of the players can play in such a game that regardless of how the other moves, the emerging plays will be wins for him. This is the question whether the game is "determined". We will formalize this by introducing the notions of strategy and winning strategy, and we will state the fundamental result that every regular game is determined; the result itself will be proved in Chapter 6.

Second, when we consider games on finite graphs these can be input for an algorithm and an obvious question to ask is if one can effectively (and maybe efficiently) determine which of the two players wins the game. This question will be answered in Chapter 7; the complexity of determining the winner heavily depends on the type of the game (the winning condition) one is interested in.

Third, it is not only interesting to know who wins a game, but also how a winning strategy looks like. Clearly, a winning strategy will tell the player what to do next depending on the moves that have been taken thus far. We will be interested in situations where the winning strategies are simple in the sense that the next move of the player does only depend on the current vertex or on a bounded amount of information on the moves that led to the current position—we will be interested in "memoryless" or "forgetful" strategies. We will describe this formally and state the main result that for every regular game there is a forgetful winning strategy and that parity games even allow memoryless strategies.

2.4 Strategies and Determinacy

In order to be able to define formally what it means for a player to win a game, we need to introduce the notion of strategy.

2.4.1 Strategies

Let $\mathcal{A}$ be an arena as usual, $\sigma \in \{0,1\}$, and $f_\sigma \colon V^*V_\sigma \to V$ a partial function. A prefix of a play $\pi = v_0 v_1 \cdots v_l$ is said to be **conform** with f_σ if for every i with $0 \leq i < l$ and $v_i \in V_\sigma$ the function f_σ is defined at $v_0 \cdots v_i$ and we have $v_{i+1} = f_\sigma(v_0 \cdots v_i)$. Note, that this also implies $v_{i+1} \in v_i E$. A play (finite or infinite) is conform with f_σ if each of its prefixes is conform with f_σ. Now we call the function f_σ a **strategy** for Player σ on $U \subseteq V$ if it is defined for every prefix of a play which is conform with it, starts in a vertex from U, and does not end in a dead end of Player σ. When U is a singleton $\{v\}$, we say f_σ is a strategy for Player σ in v.

Let $\mathcal{G} = (\mathcal{A}, \mathrm{Win})$ be an arbitrary game with $\mathcal{A}$ as usual, and f_σ a strategy for Player σ on U. The strategy f_σ is said to be a **winning strategy** for Player σ on U if all plays which are conform with f_σ and start in a vertex from U are wins for Player σ.

Example 2.2. We use the game from Example 2.1.

When Player 1 moves from z_0 to z_0 every time the token is located on z_0, then he will win every play that visits z_0. This means, in particular, that f_1 defined by $f_1(yz_0) = z_0$ and $f_1(yz_4) = z_6$ (or $= z_1$) is a winning strategy for Player 1 on $W_1 = \{z_0, z_1\}$.

Each play that doesn't begin in z_0 or z_1, visits the vertex z_2 at some point. Player 0 should under no circumstances move the token from z_2 to z_0 because his opponent could win as described above. Hence, his only chance is to move the token from z_2 to z_4. The resulting plays will visit z_2 and z_4 infinitely often.

Player 1 should not choose vertex z_2 every time the token visits vertex z_4 because this would result in a play with suffix $(z_2 z_4)^\omega$ which is a win for Player 0. So, Player 1 should once in a while move the token from z_4 to z_6.

The situation for Player 0 in vertex z_6 is a bit more complicated. If he always decides for moving the token to z_3, then the resulting play has the form $\pi = \cdots (z_6 z_3 z_2 z_4 (z_2 z_4)^*)^\omega$ and is a loss for him. Similarly, he will loose if he always moves the token to z_5. But he is able to win if he alternates between z_3 and z_5. To sum this up, consider the function f_0 defined by

$$f_0(\pi) = \begin{cases} z_4 & \text{if } \pi \in V^* z_2 \\ z_3 & \text{if } \pi \in V^* z_5 z_2 z_4 (z_2 z_4)^* z_6 \\ z_5 & \text{if } \pi \in V^* z_3 z_2 z_4 (z_2 z_4)^* z_6 \\ z_3 & \text{if } \pi \in (V \setminus \{z_3, z_5\})^* z_6 \end{cases} . \tag{2.3}$$

This is a winning strategy for Player 0 on $W_0 = \{z_2, z_3, z_4, z_5, z_6\}$.

We say that Player σ **wins** a game $\mathcal{G}$ on $U \subseteq V$ if he has a winning strategy on U.

Example 2.3. In the game from Examples 2.1 and 2.2, Player 1 wins on $\{z_0, z_1\}$ whereas Player 0 wins on $\{z_2, z_3, z_4, z_5, z_6\}$.

When $(\mathcal{G}, v_I)$ is an initialized game, we say Player σ wins it if he wins $\mathcal{G}$ on the singleton set $\{v\}$.

Clearly, every initialized game has at most one winner:

Remark 2.4. For any game $\mathcal{G}$, if Player 0 wins $\mathcal{G}$ on U_0 and Player 1 wins $\mathcal{G}$ on U_1, then $U_0 \cap U_1 = \emptyset$.

Exercise 2.1. Proof the above remark by contradiction.

Given a game $\mathcal{G}$, we define the **winning region** for Player σ, denoted $W_\sigma(\mathcal{G})$ or W_σ for short, to be the set of all vertices v such that Player 0 wins $(\mathcal{G}, v)$. Clearly:

Remark 2.5. For any game $\mathcal{G}$, Player σ wins $\mathcal{G}$ on $W_\sigma(\mathcal{G})$.

Exercise 2.2. Proof the above remark by showing that if $\mathcal{U}$ is a family of sets of vertices such that Player σ wins on each element $U \in \mathcal{U}$, then Player σ wins on $\bigcup_{U \in \mathcal{U}} U$.

2.4.2 Transforming Winning Conditions

In the previous chapter, we have seen how acceptance conditions for ω-automata can be transformed into one another. The same can be done with games. This will be explained in this section.

We first note:

Remark 2.6. For every regular game $(\mathcal{A}, \chi, \mathrm{Acc})$ there exists a Muller winning condition Acc' such that $(\mathcal{A}, \chi, \mathrm{Acc})$ and $(\mathcal{A}, \chi, \mathrm{Acc}')$ have the same winning regions.

The main result says that it is enough to consider parity games. Therefore, parity games are of our interest in the whole volume.

Theorem 2.7. *For every Muller game $(\mathcal{A}, \chi, \mathcal{F})$ there exists a parity game $(\mathcal{A}', \chi', \mathrm{Acc}')$ and a function $r\colon V \to V'$ such that for every $v \in V$, Player σ wins $((\mathcal{A}, \chi, \mathcal{F}), v)$ if and only if Player σ wins $((\mathcal{A}', \chi', \mathrm{Acc}'), r(v))$.*

Proof. The proof will be similar to the transformation of Muller conditions in Rabin conditions for ω-automaton in the previous chapter: We modify the LAR memory with hit position from Transformation 1.20 to contain colours instead of vertices because the acceptance condition for our games was defined for the colour sequence. But we have to keep track of the visited vertices too. This is done in a product construction. We will see that the constructed Rabin condition can be rewritten as Rabin chain or max-parity condition.

Let $(\mathcal{A}, \chi, \mathcal{F})$ be a Muller game, C the (finite) set of colours, and a marker $\natural \notin C$, a symbol not occurring in C. Now set our LAR memory to

$$\widetilde{C} := \{\, w \in (C \cup \{\natural\})^* \mid |w| \geq 2 \land |w|_\natural = 1 \land \forall a \in C (|w|_a \leq 1)\,\} \ . \tag{2.4}$$

$\widetilde{C}$ is the set of all words w over the alphabet $C \cup \{\natural\}$ where $\natural$ and at least one colour are infixes of w and each colour appears at most once.

Now we can define our game $(\mathcal{A}', \chi', \mathrm{Acc}')$. As vertices we choose

$$V' := V'_0 \cup V'_1 \text{ with } V'_0 := V_0 \times \widetilde{C} \text{ and } V'_1 := V_1 \times \widetilde{C} \ . \tag{2.5}$$

The set of edges is given by

$$E' := \left\{ \left((v,q),(v',\varphi(v',q))\right) \,\middle|\, v \in V,\ \ v' \in vE,\ \ q \in \widetilde{C} \right\} \tag{2.6}$$

where $\varphi\colon V \times \widetilde{C} \to \widetilde{C}$ is the memory update function that deletes the marker, replaces the colour $c := \chi(v')$ of the given vertex v' by the marker and finally appends c. Formally, φ is defined as

$$\varphi(v',q) := \begin{cases} x\natural yzc & \text{if } q = xcy\natural z \\ xy\natural zc & \text{if } q = x\natural ycz \\ qc & \text{else } (c \text{ is not an infix of } q) \end{cases} \tag{2.7}$$

for each $v' \in V$ and each $q \in \widetilde{C}$ with $c := \chi(v')$. The function that transforms the initial vertex can be set to

$$r(v) := (v, \natural\chi(v)) \ . \tag{2.8}$$

The new colouring function $\chi'\colon V' \to \omega$ is defined by

$$\chi'(v, x\natural y) := \begin{cases} 2 * |y| - 1 & \text{if } \{\, c \in C \mid c \text{ infix of } y \,\} \notin \mathcal{F} \\ 2 * |y| & \text{otherwise} \end{cases} \ . \tag{2.9}$$

We conclude the description of the construction by declaring Acc' to be a max-parity condition.

Now we have to prove the correctness of this construction which is similar to Lemma 1.21 in the previous chapter. Let $\pi = v_0 v_1 \cdots \in V^\omega$ be an infinite play in $\mathcal{A}$. The corresponding play π' in $\mathcal{A}'$ is uniquely determined: The projection onto the first component $p_1(\pi') = \pi$ is our original play, and the second component is $p_2(\pi') = q_0 q_1 \ldots \in \widetilde{C}^\omega$ with $q_i = x_i \natural y_i$ defined by $q_0 := \natural\chi(v_0)$ and $q_{i+1} := \varphi(v_{i+1}, q_i)$ for each $i \in \omega$. Let $F := \mathrm{Inf}(\chi(\pi))$ be the set of infinitely often visited colours in the play π. Hence, from some point $j \in \omega$ on the marker $\natural$ stays within the last $|F| + 1$ positions: $\forall i \geq j\ \ |y_i| \leq |F|$. Second, the marker must infinitely often occur in position $|F| + 1$, positions numbered from right to left, because each colour from F is infinitely often moved to the end. That is, $\{\, k \geq j \mid |y_k| = |F| \text{ and } y_k \text{ forms the set } F \,\}$ is infinite. Thus, by construction of χ', we have that the highest colour visited infinitely often in π' has the even value $2 \cdot |F|$ if $F \in \mathcal{F}$ and the odd value $2 \cdot |F| - 1$ otherwise. For finite plays, the situation is even simpler.

In summary, a play π is winning for Player 0 in $\mathcal{A}$ if and only if π' is winning for him in $\mathcal{A}'$. Conversely, every play π' in $\mathcal{A}'$ starting in a vertex $r(v)$ corresponds to a play π in $\mathcal{A}$, for which the same holds. So, Player 0 wins the initialized game $(\mathcal{A}, v)$ if and only if he wins $(\mathcal{A}', r(v))$. $\square$

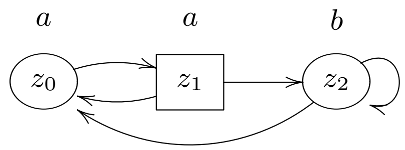

Fig. 2.2. Example for the reduction

Example 2.8. Let $\mathcal{A}$ be the arena in Figure 2.2, and $\mathcal{F} = \{\{b\}\}$ a Muller acceptance condition. The example play $\pi = z_1 z_2 z_0 z_1 z_2^\omega$ is winning for Player 0. The winning regions are $W_0 = \{z_2\}$ and $W_1 = \{z_0, z_1\}$. The constructed max-parity game $\mathcal{A}'$ is presented in Figure 2.3. We get

$$\pi' = (z_1, \natural a)(z_2, \natural ab)(z_0, \natural ba)(z_1, b\natural a)(z_2, \natural ab)(z_2, a\natural b)^\omega \tag{2.10}$$

with the colouring $\chi'(\pi') = 133132^\omega$ which is winning for Player 0 too. The winning region W_0' for Player 0 is the set of all vertices with z_2 in the first component. W_1' is the complement of W_0'.

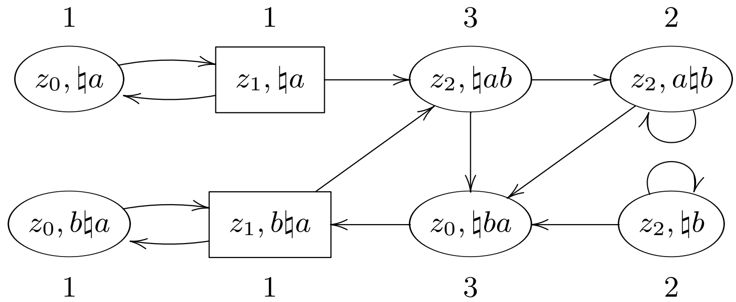

Fig. 2.3. Constructed max-parity game

2.4.3 Determinacy

In all of our example games, the winning regions for Player 0 and Player 1 partition the set of vertices of the game. When a game has this property, we will say it is **determined**.

Martin (see, e. g., [119], [95]) showed that every game with a Borel type winning set is determined. In Chapter 6, we will show the following special case of Martin's theorem.

Theorem 2.9. *Every parity game is determined.*

Together with Theorem 2.7, the previous theorem implies:

Corollary 2.10. *Every regular game is determined.*

2.4.4 Forgetful and Memoryless Strategies

The objective of this section is to introduce some notions that help to explain how complex a winning strategy is.

As a motivation, consider the game from Example 2.1 again. We argued that in order to win it is necessary for Player 0 to alternate between moving the token to z_3 and z_5 when it is on z_6. More precisely, it is necessary not to stick to one of the two vertices from some point onwards. This means that Player 0 has to remember at least one bit, namely whether he moved to z_3 or z_5 when the token was on z_6 the last time. But from our argumentation, it is also clear that it is not necessary to remember more than that. In other words, a finite memory is sufficient for Player 0 to carry out his strategy. We will say Player 0 has a forgetful strategy. The situation is much easier for Player 1. He does not need to remember anything; he simply moves to z_0 every time the token is on z_0. We will say Player 1 has a memoryless strategy.

Let $\mathcal{G}$ be a game as usual. A strategy f_σ is said to be **finite memory** or **forgetful** if there exists a finite set M, an element $m_I \in M$, and functions $\delta\colon V \times M \to M$ and $g\colon V \times M \to V$ such that the following is true. When $\pi = v_0 v_1 \cdots v_{l-1}$ is a prefix of a play in the domain of f_σ and the sequence $m_0, m_1, \ldots, m_l$ is determined by $m_0 = m_I$ and $m_{i+1} = \delta(v_i, m_i)$, then $f_\sigma(\pi) = g(v_l, m_l)$.

Forgetful strategies that don't need memory at all, that is, where one can choose M to be a singleton, are called **memoryless** or **positional**. Observe that a memoryless strategy f_σ has the property that whenever f_σ is defined for πv and $\pi' v$, then $f_\sigma(\pi v) = f_\sigma(\pi' v)$. This allows us to view memoryless strategies as partial functions $V_\sigma \to V$, and, for ease in notation, we will often use this representation.

Example 2.11. In Example 2.2, the strategy f_1 for Player 1 is memoryless. To see this, observe that we can choose M to be a singleton, say $M = \{m\}$, and set $g(z_0, m) = z_0$ and $g(z_3, m) = g(z_4, m) = z_2$. So, Player 1 has a memoryless winning strategy on $W_1 = \{z_0, z_1\}$. Using the simplified notation, we could write $f_1(z_0) = z_0$ and $f_1(z_3) = f_1(z_4) = z_2$.

Player 0 needs to store which one of the colours 3 (occurring on vertex z_3) and 4 (on vertex z_5) he visited last. This can be done with a memory $M = \{3, 4\}$. More precisely, one can choose $m_I = 3$,

$$\delta(v,m) = \begin{cases} 3 & \text{if } v = z_3 \\ 4 & \text{if } v = z_5 \\ m & \text{otherwise} \end{cases} . \tag{2.11}$$

and

$$g(v,m) = \begin{cases} z_4 & \text{if } v = z_2 \\ z_3 & \text{if } v = z_6 \text{ and } m = 4 \\ z_5 & \text{if } v = z_6 \text{ and } m = 3 \end{cases} . \tag{2.12}$$

Thus, Player 0 has a forgetful winning strategy on $W_0 = \{z_2, z_3, z_4, z_5, z_6\}$.

In Example 2.2, we already stated that Player 0 must not move from z_6 to the same successor every time he visits z_6. So, Player 0 can't have a memoryless winning strategy.

We say that Player σ wins a game $\mathcal{G}$ **forgetful** when he has a forgetful strategy for each point of his winning region. Accordingly, it is defined what it means to win **with finite memory**, **memoryless**, and **positional**.

Exercise 2.3. Give an example for a game $\mathcal{G}$ such that Player 0 wins forgetful on each $\{v\}$ for $v \in W_0$, but he has no forgetful winning strategy on W_0. Can you give an example where $\mathcal{G}$ is regular?

In exercise 2.2, the reader was asked to show that if U is some set of vertices such that Player σ wins a given game $\mathcal{G}$ on every element of U, then he wins $\mathcal{G}$ on U. This is easy to see. In Exercise 2.3, the reader is asked to provide an example that shows that the corresponding statement is not true for forgetful strategies. However, a corresponding statement is true for memoryless strategies under a certain condition:

Lemma 2.12. *Let* $\mathcal{G} = (\mathcal{A}, \mathrm{Win})$ *be any game with countable vertex set* V*,*

$$V^*\mathrm{Win} \subseteq \mathrm{Win} \qquad \text{and} \qquad \mathrm{Win}/V^* \subseteq \mathrm{Win}, \tag{2.13}$$

where $\mathrm{Win}/V^* := \{\, \eta \in V^\omega \mid \exists w \in V^* \text{ with } w\eta \in \mathrm{Win} \,\}$ *is the set of all suffixes of* Win*. Let* U *be a set of vertices such that Player* σ *has a memoryless winning strategy for each element from* U*. Then Player* σ *has a memoryless winning strategy on* U*.*

Before we turn to the proof observe that the two conditions on the winning set are satisfied in every regular game: A prefix of a winning play can be substituted by any other finite word; the set of infinitely often visited colours stays the same.

Proof. The proof uses the axiom of choice. For every $u \in U$, let $f_\sigma^u \colon V_\sigma \to V$ be a partial function which is a memoryless winning strategy for Player σ on u. Without loss of generality, we assume that for every $u \in U$ the domain of f_σ^u, denoted D_u, is minimal with respect to set inclusion.

Let $<$ be a well-ordering on U (therefore we choose V to be countable) and $D := \bigcup_{u \in U} D_u$. We have to define a memoryless winning strategy $f_\sigma \colon D \to V$.

For each $v \in D$, let $u(v)$ be the minimal vertex in U (with respect to the well-ordering) with $v \in D_{u(v)}$, and set $f_\sigma(v) := f_\sigma^{u(v)}(v)$. Clearly, f_σ is well defined and memoryless. We have to show that f_σ is a winning strategy on U.

Assume $\pi = v_0 v_1 \cdots$ is a play starting in U and conform with f_σ. In each σ-vertex v_j of the play π, Player σ has to choose the strategy $f_\sigma^{u(v_j)}$. Let i be such that $u(v_i)$ is minimal (with respect to the well-ordering) in the set $\{\, u(v_j) \mid j \in \omega \text{ and } v_j \in D \,\}$. Then, from this moment i on, the strategy f_σ follows the strategy $f_\sigma^{u(v_i)}$. The domain $D_{u(v_i)}$ was minimal with respect to set inclusion, thus, the play $v_i v_{i+1} \cdots$ is a suffix of a play that starts in $u(v_i)$, visits v_i, and is conform to $f_\sigma^{u(v_i)}$. Hence, $\pi \in V^*(\mathrm{Win}/V^*) \subseteq \mathrm{Win}$ by our two conditions, which completes the proof. □

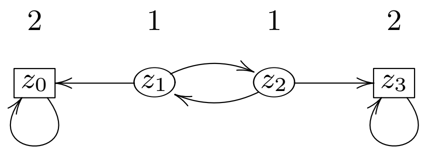

Fig. 2.4. Example for the construction of a memoryless strategy

Example 2.13. Let $\mathcal{A}$ be the max-parity game in Figure 2.4. Clearly, Player 0 wins on each $v \in U = \{z_1, z_2\}$, i. e. with the memoryless strategies

- $f_0^{z_1}(z_1) = z_2$ and $f_0^{z_1}(z_2) = z_3$,
- $f_0^{z_2}(z_2) = z_1$ and $f_0^{z_2}(z_1) = z_0$.

To find a memoryless strategy on U, Player 0 can not set $f_0(z_1) = f_0^{z_1}(z_1)$ and $f_0(z_2) = f_0^{z_2}(z_2)$ because this yields an infinite loop in z_1 and z_2 which is a loss for him. If $z_1 < z_2$ in the well-ordering of U, then we get $f_0 \equiv f_0^{z_1}$. This is a memoryless winning strategy on U.

In Theorem 6.6 in Chapter 6 we will show the following.

Theorem 2.14. *In every parity game, both players win memoryless. This is called* ***memoryless determinacy of parity games****.*

From this, together with the construction in the proof of Theorem 2.7, we can conclude:

Corollary 2.15. *In every regular game, both players win forgetful. Analogously, this is called* ***forgetful*** *or* ***finite memory determinacy of regular games****.*

Proof. Let $(\mathcal{A}, \chi, \mathcal{F})$ be a Muller game, $\mathcal{A}'$ the max-parity game as constructed in the proof of Theorem 2.7, and $V‘ = V \times \widetilde{C}$ the set of vertices of $\mathcal{A}'$ with $\widetilde{C}$ defined in Equation 2.4. The memoryless determinacy of parity games yields memoryless winning strategies f'_0 and f'_1 on the winning regions W'_0 and W'_1 with $W'_0 \cup W'_1 = V'$.

Now the observations in the proof of Theorem 2.7 allow us to construct forgetful strategies in $\mathcal{A}$. The winning regions are $W_\sigma = \{ v \in V \mid (v, \natural\chi(v)) \in W'_\sigma \}$ for $\sigma \in \{0, 1\}$. We can use the finite memory $M = \widetilde{C}$ for both strategies. As initial memory state of $(\mathcal{A}, v)$ we choose $m_I = \natural\chi(v)$. The memory update function δ is equal to φ from Equation 2.7. The forgetful strategies g_0 and g_1 are defined by

$$g_\sigma(v, q) := f'_\sigma((v, q)) \tag{2.14}$$

for $\sigma \in \{0, 1\}$, $v \in V_\sigma \cap W_\sigma$, and $q \in \widetilde{C}$.

Clearly, these strategies are forgetful winning strategies because g_σ simulates f'_σ. □

Note that the initial memory state in the previous construction could be chosen arbitrarily.

Exercise 2.4. Using the results from the previous chapter, determine how much memory is sufficient and necessary to win Rabin and Muller games.

Theorem 2.14 states that parity games enjoy memoryless determinacy, that is, winning strategies for both players can be chosen memoryless. It is easy to show that in certain Muller games both players need memory to win. In between, we have Rabin and Streett conditions. For those, one can actually prove that one of the two players always has a memoryless winning strategy, but we will not carry out the proof in this volume.

Theorem 2.16. *In every Rabin game, Player 0 has a memoryless winning strategy on his winning region. Symmetrically, in every Streett game, Player 1 has a memoryless strategy on his winning region.*

This theorem can also be applied to certain Muller automata on the grounds of the following observation. A Muller condition $(\mathcal{F}_0, \mathcal{F}_1)$ can be rephrased as Rabin condition if and only if $\mathcal{F}_1$ is closed under union.

Example 2.17. We got a memoryless strategy for Player 1 in our Example 2.11. His winning condition $\mathcal{F}_1$ is expressible as Rabin condition: $\{(\{3\},\{4\}),(\{4\},\{3\}),(\{1\},\{2\})\}$. He wins a play if it loops, for instance, finitely often through one of the colours 3 or 4 and infinitely often through the other colour. Note that the winning condition cannot be rephrased as a parity condition, that is, Rabin chain condition (on the same graph).

2.5 Solving Games with Simple Winning Conditions

In this section, we prove special instances of Corollaries 2.10 and 2.15 and Theorem 2.14.

2.5.1 Reachability Games and Attractors

For a start, we consider games which do not really fit into the framework that we have used thus far. Given an arena $\mathcal{A} = (V_0, V_1, E)$ and a set $X \subseteq V$ the **reachability game** $R(\mathcal{A}, X)$ is the game in which a play π (be it finite or infinite) is winning for Player 0 if some vertex from X or a dead end belonging to Player 1 occurs in π. This is different from the games we have studied so far because a dead end for Player 0 does not need to be a loosing position for him. Strategies for reachability games are defined as before, but with the difference that a strategy for Player 0 does not need to be defined for arguments that end in a vertex from X.

Proposition 2.18. *Reachability games enjoy memoryless determinacy.*

Proof. The proof is constructive in the sense that on finite graphs it can be immediately turned into an algorithm which computes the winning regions and the memoryless winning strategies.

Let $\mathcal{A}$ be an arena as usual and $X \subseteq V$. The winning region for Player 0 in $R(\mathcal{A}, X)$ and a memoryless winning strategy for Player 0 are defined inductively. In the inductive step, we use the function $pre\colon \mathscr{P}(V) \to \mathscr{P}(V)$ defined by

$$pre(Y) = \{ v \in V_0 \mid vE \cap Y \neq \emptyset \} \cup \{ v \in V_1 \mid vE \subseteq Y \} \tag{2.15}$$

Inductively, we set $X^0 = X$,

$$X^{\nu+1} = X^\nu \cup pre(X^\nu) \tag{2.16}$$

for all ordinals ν, and

$$X^\xi = \bigcup_{\nu<\xi} X^\nu \tag{2.17}$$

for each limit ordinal ξ. Let ξ be the smallest ordinal such that $X^\xi = X^{\xi+1}$. We claim that $W := X^\xi$ is Player 0's winning region. Clearly, for every $v \in W \setminus X$ there exists a unique ordinal $\xi_v < \xi$ such that $v \in X^{\xi_v+1} \setminus X^{\xi_v}$. By the above definition, we furthermore know that for every $v \in W \cap V_0 \setminus X$ there exists $v' \in vE$ such that $v' \in X^{\xi_v}$. We set $f_0(v) = v'$ and claim that f_0 is a memoryless strategy for Player 0 on W. This can be easily proved by transfinite induction: One shows that f_0 is winning for Player 0 on X^ν for every $\nu \leq \xi$. Hence, $W \subseteq W_0$.

On the other hand, let $W' = V \setminus W$ and assume $v \in W'$. Then $v \notin X$. If v is a dead end, it must be a dead end of Player 0 because all dead ends of Player 1 belong to X^1. But, on a dead end belonging to Player 0, Player 1 wins immediately. If v is no dead end and belongs to V_0, we have $v' \notin W$ for every $v' \in vE$ because otherwise v would belong to W. Similarly, if v is no dead end and belongs to V_1, there exists $v' \in vE$ such that $v' \notin W$ because otherwise v would belong to W. If we set $f_1(v) = v'$ in this case, then f_1 is clearly a memoryless strategy for Player 1. Every play conform with this strategy and starting in W' has the property that all its vertices belong to W'. Since W' does not contain vertices from X or dead ends of Player 1 this play must be winning for Player 1. Hence, f_1 is a winning strategy for Player 1 on W' and $V \setminus W = W' \subseteq W_1$, that is, $W_0 = W$ and $W_1 = V \setminus W$. □

The winning region of Player 0 in a reachability game $R(\mathcal{A}, X)$ is denoted $\mathrm{Attr}_0(\mathcal{A}, X)$ and called **0-attractor** of the set X in the arena $\mathcal{A}$. A memoryless winning strategy f_0 as described in the above prove is called a corresponding **attractor strategy** for Player 0. 1-attractor and attractor strategy for Player 1 are defined symmetrically, simply by exchanging V_0 and V_1 in the arena.

Exercise 2.5. Let $\mathcal{A}$ be an arbitrary arena, $X \subseteq V$, and $a_X\colon \mathscr{P}(V) \to \mathscr{P}(V)$ the function defined by

$$a_X(U) := X \cup pre(U) \ . \tag{2.18}$$

Show that a_X is monotone with respect to set inclusion and that $\mathrm{Attr}_0(\mathcal{A}, X)$ is the least fixed point of a_X.

Exercise 2.6. Show that in a finite arena with n vertices and m edges the attractor of any set can be computed in time $O(m+n)$.

Exercise 2.7. Let $\mathcal{A}$ be an arena and $Y = V \setminus \mathrm{Attr}_\sigma(\mathcal{A}, X)$ for some $X \subseteq V$. Show that σ cannot escape Y in the sense that $vE \subseteq Y$ for every $v \in Y \cap V_\sigma$ and $vE \cap Y \neq \emptyset$ for every $v \in Y \cap V_{\overline{\sigma}}$.

This exercise motivates the following definition. A **σ-trap** is a subset $Y \subseteq V$ such that $vE \subseteq Y$ for every $v \in Y \cap V_\sigma$ and $vE \cap Y \neq \emptyset$ for every $v \in Y \cap V_{\overline{\sigma}}$. A function which picks for every $v \in Y \cap V_{\overline{\sigma}}$ a vertex $v' \in vE \cap Y$ is called a **trapping strategy** for Player $\overline{\sigma}$.

Remark 2.19. The complement of a σ-attractor is a σ-trap.

The above remark tells us that, without loss of generality, we can assume that arenas have no dead ends. Let $(\mathcal{A}, \mathrm{Acc})$ be an arbitrary game with $\mathcal{A} = (V_0, V_1, E)$. For $\sigma \in \{0, 1\}$, we set $U_\sigma = \mathrm{Attr}_\sigma(\mathcal{A}, \emptyset)$. Then Player σ wins $(\mathcal{A}, \mathrm{Acc})$ on U_σ memoryless. Now, let $V_0' = V_0 \setminus (U_0 \cup U_1)$ and $V_1' = V_1 \setminus (U_0 \cup U_1)$ and consider the arena $\mathcal{A}' = (V_0', V_1', E \cap ((V_0' \cup V_1') \times (V_0' \cup V_1')))$. Clearly, $\mathcal{A}'$ does not have any dead end. Further, for every $v \in V_0' \cup V_1'$, Player 0 wins $(\mathcal{A}', \mathrm{Acc}, v)$ iff he wins $(\mathcal{A}, \mathrm{Acc}, v)$ and, symmetrically, Player 1 wins $(\mathcal{A}', \mathrm{Acc}, v)$ iff he wins $(\mathcal{A}, \mathrm{Acc}, v)$. More specifically, winning strategies for $(\mathcal{A}', \mathrm{Acc})$ can be used in $(\mathcal{A}, \mathrm{Acc})$.

Exercise 2.8. Work out the details of the above argument.

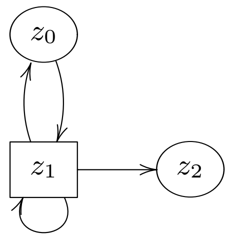

Fig. 2.5. Dead end strategy for Player 1

Example 2.20. In the game depicted in Figure 2.5, Player 1 may prevent an infinite play by moving the token to z_2. This is a dead end for Player 0 and Player 1 wins.

2.5.2 1-acceptance

Using what we have proved about reachability games, we can now easily solve 1-games.

Proposition 2.21. *1-games enjoy memoryless determinacy.*

Proof. Let $\mathcal{G} = (\mathcal{A}, \chi, F)$ and define Y and V' by $Y = \mathrm{Attr}_1(\mathcal{G}, \emptyset)$ and $V' = V \backslash Y$. Let $\mathcal{A}' = (V_0 \cap V', V_1 \cap V', E \cap (V' \times V'))$. Observe that $\mathcal{A}'$ does not contain any dead end of Player 0. We claim that $W := \mathrm{Attr}_0(\mathcal{A}', \chi^{-1}(F))$ is the winning region of Player 0 in $\mathcal{G}$.

Clearly, Y is a subset of the winning region of Player 1. Further, $W \subseteq W_0$, because on this set Player 0 can force the game into a dead end of Player 1 or a vertex in $\chi^{-1}(F)$ and go on forever because $\mathcal{A}'$ does not contain any dead end of Player 0. Remember that V' is a 1-trap, that is, Player 1 cannot escape V'. And on both sets, Y and W we have memoryless winning strategies (attractor and trapping strategies) for the respective players. It is now sufficient to show that Player 1 has a memoryless winning strategy on $Z := V' \setminus W$. Since Z is a 0-trap of $\mathcal{A}'$, Player 1 can use his trapping strategy and the token will then stay in Z forever or stay in Z until it is moved to a vertex in Y, which is winning for Player 1 anyway. □

Exercise 2.9. Show that for finite arenas, the winning regions of 1-games can be computed in time $O(m + n)$. (See also Exercise 2.6.)

2.5.3 Büchi Acceptance

Obviously, Büchi games can viewed as parity games. So memoryless determinacy follows from memoryless determinacy of parity games, which will be proved in Chapter 6. Nevertheless, we give a straightforward proof along the lines of the proofs that we have seen in the previous two subsections.

Theorem 2.22. *Büchi games enjoy memoryless determinacy.*

Proof. Like in the other solutions, we first describe how to construct the winning region for Player 0 in a Büchi game $(\mathcal{A}, \chi, F)$.

We set $Y = \chi^{-1}(F)$, and define inductively:

$$Z^0 = V \ , \tag{2.19}$$

$$X^\xi = \mathrm{Attr}_0(\mathcal{A}, Z^\xi) \ , \tag{2.20}$$

$$Y^\xi = pre(X^\xi) \ , \tag{2.21}$$

$$Z^{\xi+1} = Y^\xi \cap Y \ , \tag{2.22}$$

$$Z^\xi = \bigcup_{\nu<\xi} Z^\nu \ , \tag{2.23}$$

where the last equation only applies to limit ordinals ξ. Let ξ be the least ordinal ≥ 1 such that $Z^\xi = Z^{\xi+1}$. We claim $W := \mathrm{Attr}_0(\mathcal{A}, Z^\xi)$ is the winning region of Player 0.

To prove $W \subseteq W_0$, we describe a memoryless winning strategy f_0 for Player 0 on W. For every $v \in V_0 \cap Z^\xi$, there exists $v' \in vE \cap \mathrm{Attr}_0(\mathcal{A}, Z^\xi)$ and we set $f_0(v) = v'$. For every other $v \in V_0 \cap W$, we know $v \in \mathrm{Attr}_0(\mathcal{A}, Z^\xi)$, and thus we set $f_0(v)$ to the value of a respective attractor strategy. Now, the following is easy to see. First, if a finite play starting in W is conform with f_0, then it ends in a dead of Player 1, which means Player 0 wins. Second, if an infinite play starting in W is conform with f_0 it eventually reaches Z^ξ and from this point onwards it will reach Z^ξ over and over again. But since $Z^\xi \subseteq Y$ (this is because $\xi \geq 1$), the play will be winning for Player 0.

To prove that $W_0 = W$, we argue that Player 1 has a memoryless winning strategy on $W' := V \setminus W$. The winning strategy is defined as follows. For every $v \in W'$ there exists a least ν such that $v \in X^\nu \setminus X^{\nu+1}$. (Note that $X^0 = V$ and $X^{\nu'} \subseteq X^{\nu''}$ for all ordinals ν' and ν'' with $\nu'' < \nu'$.) Since $X^{\nu+1}$ is a 0-attractor, $V \setminus X^{\nu+1}$ is a 0-trap. We set $f_1(v)$ to the value of a trapping strategy for Player 1 if $v \notin Y$. Otherwise, it follows that $v \notin pre(X^\nu)$, and thus, there exists some $v' \in vE \cap V \setminus X^\nu$. We set $f_1(v) = v'$. By induction on ν, it is now easy to show that f_1 is a winning strategy for Player 1 on $V \setminus X^\nu$. It follows that f_1 is a winning strategy on W'. □

Exercise 2.10. Show that for a finite arena, the winning regions of a Büchi game can be computed in time $O(n(m + n))$.