Infinite Games

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Build Correct HW/SW Systems

► Use logic to specify correctness properties, e.g.:
  ► every job sent to the printer is eventually printed
  ► two jobs do not overlap (only one job is printed at a time)
  ► a job that is canceled will be interrupted

These are conditions on infinite sequences (system runs), and can be specified by automata and logical formulas.
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These are conditions on infinite sequences (system runs), and can be specified by automata and logical formulas.

- Given a logical specification, we can do either:
  - Verification: prove that a given system satisfies the specification
  - Synthesis: build a system that satisfies the specification
Example: Elevator

- **Aim:** build controller that moves elevator of 10 floor building
- **Environment:** Passengers pressing buttons to (1) call elevator and (2) request floor
- **System state:**
  1. Set of requested floor numbers: \( \{0, 1\}^{10} \)
  2. Current position of lift: \( \{1, \ldots, 10\} \)
  3. Indicator whose turn is next (assuming lift and passengers act in alternation) \( \{0, 1\} \)
Infinite Games

Two players:

1. Controller is Player 0
2. Passengers are Player 1

A play of a game is an infinite sequence of states of elevator transition system, where the two players choose moves alternatively.

How does the transition system look like?

- State space: $\{0, 1\}^{10} \times \{1, \ldots, 10\} \times \{0, 1\}$
- Transitions:
  - Player 0: \((r_1 \ldots r_{10}, j, 0) \rightarrow [r'_1 \ldots r'_{10}, j', 1]\) s.t. $r_j = 1$, $r'_j = 0$ and $\forall i \neq j r_i = r'_i$. Actions: open/closes doors and move lift
  - Player 1: \([r_1 \ldots r_{10}, j, 1] \rightarrow (r'_1 \ldots r'_{10}, j', 0)\) s.t. $j = j'$, $\forall i : r_i \leq r'_i$. Actions: request floors
Desired Properties

- Every requested floor is eventually reached
- Floors along the way are served if requested
- If no floor is requested, elevator goes to ground floor
- ...

These are conditions on infinite sequences!

Player 0 (controller) wins the play if all conditions are satisfied independent of the choices Player 1 makes. This corresponds to finding a winning strategy for Player 0 in an infinite game.
Our Aim

Solution of the Synthesis Problem

1. Decide whether there exists such a winning strategy - Realizability Problem

2. If “yes”, then construct the system - Synthesis Problem

Main result:
The synthesis problem is algorithmically solvable for finite-state systems with respect to specifications given as $\omega$-automata or linear-time temporal logic.

Other: Nicer and more intuitive proofs for logics over trees
Outline

1. Terminology

2. Games
   2.1 Reachability games
   2.2 Buchi games
   2.3 Obligation games
   2.4 Muller games

3. About games and tree automata
Terminology
Two-player games between Player 0 and 1

An infinite game $\langle G, \phi \rangle$ consists of

- a game graph $G$ and
- a winning condition $\phi$.

$G$ defines the “playground”, in which the two players compete.
$\phi$ defines which plays are won by Player 0.
If a play does not satisfy $\phi$, then Player 1 wins on this play.
A **game graph** is a tuple \( G = \langle S, S_0, T \rangle \) where:

- \( S \) is a finite set of **states**, 
- \( S_0 \subseteq S \) is the set of **Player-0 states** (\( S_1 = S \setminus S_0 \) are the **Player-1 states**), 
- \( T \subseteq S \times S \) is a **transition relation**. We assume that each state has at least one successor.

![Game Graph Diagram]

- **Player 0**
- **Player 1**
A play is an infinite sequence of states $\rho = s_0 s_1 s_2 \cdots \in S^\omega$ such that for all $i \geq 0$ $\langle s_i, s_{i+1} \rangle \in T$.

It starts in $s_0$ and it is built up as follows:
If $s_i \in S_0$, then Player 0 chooses an edge starting in $s_i$, otherwise Player 1 picks such an edge.

Intuitively, a token is moved from state to state via edges: From $S_0$-states Player 0 moves the token, from $S_1$-states Player 1 moves the token.
Winning Condition

The winning condition describes the plays won by Player 0.
A winning condition or winning objective $\phi$ is a subset of plays, i.e., $\phi \subseteq S^\omega$.
We use logical conditions (e.g., LTL formulas) or automata theoretic acceptance conditions to describe $\phi$.
Example:

- $\Box \Diamond s$ for some state $s \in S$
- All plays that stay within a safe region $F \subseteq S$ are in $\phi$.
- Given a priority function $p : S \rightarrow \{0, 1, \ldots, d\}$, all plays in which the smallest priority visited is even.

Games are named after their winning condition, e.g., Safety game, Reachability game, LTL game, Parity game,...
Types of Games

Given a play $\rho$, we define

- $\text{Occ}(\rho) = \{ s \in S \mid \exists i \geq 0 : s_i = s \}$
- $\text{Inf}(\rho) = \{ s \in S \mid \forall i \geq 0 \exists j > i : s_j = s \}$

Given a set $F \subseteq S$,

- **Reachability Game** $\phi = \{ \rho \in S^\omega \mid \text{Occ}(\rho) \cap F \neq \emptyset \}$
- **Safety Game** $\phi = \{ \rho \in S^\omega \mid \text{Occ}(\rho) \subseteq F \}$
- **Büchi Game** $\phi = \{ \rho \in S^\omega \mid \text{Inf}(\rho) \cap F \neq \emptyset \}$
- **Co-Büchi Game** $\phi = \{ \rho \in S^\omega \mid \text{Inf}(\rho) \subseteq F \}$
We will refer to the type of a game and give $F$, $p$, or $\varphi$ instead of defining $\phi$. We will also talk about Muller and Rabin games.
Strategies

A strategy for Player 0 from state $s$ is a (partial) function

$$f : S^*S_0 \rightarrow S$$

specifying for any sequence of states $s_0, s_1, \ldots s_k$ with $s_0 = s$ and $s_k \in S_0$ a successor state $s_j$ such that $(s_k, s_j) \in T$.

A play $\rho = s_0s_1\ldots$ is compatible with strategy $f$ if for all $s_i \in S_0$ we have that $s_{i+1} = f(s_0s_1\ldots s_i)$.

(Definitions for Player 1 are analogous.)

Given strategies $f$ and $g$ from $s$ for Player 0 and 1, respectively, we denote by $G_{f,g}$ the (unique) play that is compatible with $f$ and $g$. 
Winning Strategies and Regions

Given a game \((G, \phi)\) with \(G = (S, S_0, E)\), a strategy \(f\) for Player 0 from \(s\) is called a winning strategy if for all Player-1 strategies \(g\) from \(s\), \(G_{f,g} \in \phi\) holds. Analogously, a Player-1 strategy \(g\) is winning if for all Player-0 strategies \(f\), \(G_{f,g} \not\in \phi\) holds.

Player 0 (resp. 1) wins from \(s\) if s/he has a winning strategy from \(s\).
Winning Strategies and Regions

Given a game \((G, \phi)\) with \(G = (S, S_0, E)\), a strategy \(f\) for Player 0 from \(s\) is called a \textbf{winning strategy} if for all Player-1 strategies \(g\) from \(s\), \(G_{f,g} \in \phi\) holds. Analogously, a Player-1 strategy \(g\) is winning if for all Player-0 strategies \(f\), \(G_{f,g} \notin \phi\) holds.

Player 0 (resp. 1) wins from \(s\) if s/he has a winning strategy from \(s\). The \textbf{winning regions} of Player 0 and 1 are the sets

\[
W_0 = \{ s \in S \mid \text{Player 0 wins from } s \}
\]

\[
W_1 = \{ s \in S \mid \text{Player 1 wins from } s \}
\]

Note each state \(s\) belongs at most to \(W_0\) or \(W_1\). Otherwise pick winning strategies \(f\) and \(g\) from \(s\) for Player 0 and 1, respectively, then \(G_{f,g} \in \phi\) and \(G_{f,g} \notin \phi\): Contradiction.
Questions About Games

Solve a game \((G, \phi)\) with \(G = (S, S_0, T)\):

1. Decide for each state \(s \in S\) if \(s \in W_0\).

2. If yes, construct a suitable winning strategy from \(s\).

Further interesting question:

- Optimize construction of winning strategy (e.g., time complexity)
- Optimize parameters of winning strategy (e.g., size of memory)
Example

Safety game \((G, F)\) with \(F = \{s_0, s_1, s_3, s_4\}\), i.e., \(\text{Occ}(\rho) \subseteq F\)

A winning strategy for Player 0 (from state \(s_0\) and \(s_4\)):

- From \(s_0\) choose \(s_3\) and from \(s_4\) choose \(s_3\)

A winning strategy for Player 1 (from state \(s_1\) and \(s_2\)):

- From \(s_1\) choose \(s_2\), from \(s_2\) choose \(s_4\)
Example

Safety game \((G, F)\) with \(F = \{s_0, s_1, s_3, s_4\}\), i.e., \(\text{Occ}(\rho) \subseteq F\)

A winning strategy for Player 0 (from state \(s_0\) and \(s_4\)):

- From \(s_0\) choose \(s_3\) and from \(s_4\) choose \(s_3\)

A winning strategy for Player 1 (from state \(s_1\) and \(s_2\)):

- From \(s_1\) choose \(s_2\), from \(s_2\) choose \(s_4\)

\(W_0 = \{s_0, s_3, s_4\}\), \(W_1 = \{s_1, s_2\}\)
LTL game \((G, \varphi)\) with \(\varphi = \Diamond s_0 \land \Diamond s_4\) (visit \(s_0\) and \(s_4\))

Winning strategy for Player 0 from \(s_0\):

- From \(s_0\) to \(s_3\), from \(s_3\) to \(s_4\), and from \(s_4\) to \(s_1\).

Note: this strategy is not winning from \(s_3\) or \(s_4\).

Winning strategy for Player 0 from \(s_3\):

- From \(s_3\) to \(s_0\) on first visit, otherwise to \(s_4\).
Determinacy

Recall: the winning regions are disjoint, i.e., $W_0 \cap W_1 = \emptyset$

Question: Is every state winning for some player?

A game $(G, \phi)$ with $G = (S, S_0, E)$ is called determined if $W_0 \cup W_1 = S$ holds.

Remarks:

1. We will show that all automata-theoretic games we consider here are determined.

2. There are games which are not determined (e.g., concurrent games: even/odd sum, paper-rock-scissors)
Strategy Types

In general, a strategy is a function \( f : S^+ \to S \).
(Note that sometimes we might define \( f \) only partially.)

1. Computable or recursive strategies: \( f \) is computable

2. Forgetful or finite-state strategies: \( f \) is computable with a finite-state automaton meaning that \( f \) has bounded information about the past (history).

3. Memoryless or positional strategies: \( f \) only depends on the current state of the game (no knowledge about history of play)
Positional Strategies

Given a game \((G, \phi)\) with \(G = (S, S_0, E)\), a strategy \(f : S^+ \rightarrow S\) is called **positional or memoryless** if for all words \(w, w' \in S^+\) with \(w = s_0 \ldots s_n\) and \(w' = s'_0 \ldots s'_m\) such that \(s_n = s'_m\), \(f(w) = f(w')\) holds.

A positional strategy for Player 0 is representable as

1. a function \(f : S_0 \rightarrow S\)

2. a set of edges containing for every Player-0 state \(s\) exactly one edge starting in \(s\) (and for every Player-1 state \(s'\) all edges starting in \(s'\))
Finite-state Strategies

A strategy automaton over a game graph $G = (S, S_0, E)$ is a finite-state machine $A = (M, m_0, \delta, \lambda)$ (Mealy machine) with input and output alphabet $S$, where

- $M$ is a finite set of states (called memory),
- $m_0 \in M$ is an initial state (the initial memory content),
- $\delta : M \times S \rightarrow M$ is a transition function (the memory update fct),
- $\lambda : M \times S \rightarrow S$ is a labeling function (called the choice function).
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- $\lambda : M \times S \rightarrow S$ is a labeling function (called the choice function).

The strategy for Player 0 computed by $A$ is the function

$$f_A(s_0 \ldots s_k) := \lambda(\delta(m_0, s_0 \ldots s_{k-1}), s_k) \text{ with } s_k \in S_0$$

and the usual extension of $\delta$ to words: $\delta(m_0, \epsilon) = m_0$ and $\delta(m_0, s_0 \ldots s_k) = \delta(\delta(m_0, s_0 \ldots s_{k-1}), s_k)$. Any strategy $f$, such that there exists an $A$ with $f_A = f$, is called finite-state strategy.
Recall Example

Objective: visit $s_0$ and $s_4$, i.e, $\{s_0, s_4\} \subseteq \text{Occ}(\rho)$

Winning strategy for Player 0 from $s_0$, $s_3$ and $s_4$:

- From $s_0$ to $s_3$, from $s_4$ to $s_3$, and from $s_3$ to $s_0$ on first visit, otherwise to $s_4$. $s_0/s_3$
  $s_1/-$
  $s_2/-$
  $s_4/s_3$

- From $s_0$ to $s_3$, from $s_4$ to $s_3$, and from $s_3$ to $s_0$ on first visit, otherwise to $s_4$. $s_0/s_3$
  $s_1/-$
  $s_2/-$
  $s_3/s_4$
  $s_4/s_3$
Extended Game

\[
s_0 / s_3 \\
s_1 / \_ \\
s_2 / \_ \\
s_4 / s_3
\]

\[
s_0 / s_3 \\
s_1 / \_ \\
s_2 / \_ \\
s_3 / s_4 \\
s_4 / s_3
\]
Extended Game
Extended Game
Extended Game

Note: the strategy in the extended game graph is memoryless.
Reachability and Safety Games
**Theorem**

Given a reachability game \((G, F)\) with \(G = (S, S_0, E)\) and \(F \subseteq S\), then the winning regions \(W_0\) and \(W_1\) of Player 0 and 1, respectively, are computable, and both players have corresponding memoryless winning strategies.

**Proof.**

Define

\[
\text{Attr}^i_0(F) := \{ s \in S \mid \text{Player 0 can force a visit from } s \text{ to } F \text{ in less than } i \text{ moves} \}
\]
Force Visit in Next Step

Given a set of states, compute the set of states \( \text{ForceNext}_0(F) \) from which of Player 0 can force to visit \( F \) in the next step. I.e., for each state \( s \in \text{ForceNext}_0(F) \) Player 0 can fix a strategy s.t. all plays starting in \( s \) visit \( F \) in the first step.

\[
\text{ForceNext}_0(F) = \{ s \in S_0 \mid \exists s' \in S : (s, s') \in E \land s' \in F \} \cup \\
\{ s \in S_1 \mid \forall s' \in S : (s, s') \in E \rightarrow s' \in F \}
\]
Computing the Attractor

Construction of $\text{Attr}_0^i(F)$:

$$\text{Attr}_0^0(F) = F$$
$$\text{Attr}_0^{i+1}(F) = \text{Attr}_0^i(F) \cup \text{ForceNext}_0(\text{Attr}_0^i(F))$$
Example
Example

\[ \text{Attr}_0^0 = \{ s_3, s_4 \} \]
Example

$\text{Attr}_0 = \{s_3, s_4\}$
$\text{Attr}_1 = \{s_0, s_3, s_4\}$
Example

Attr\textsubscript{0} = \{s_3, s_4\}
Attr\textsubscript{1} = \{s_0, s_3, s_4\}
Attr\textsubscript{2} = \{s_0, s_3, s_4, s_7\}
Example

\[ \text{Attr}_0^0 = \{s_3, s_4\} \]
\[ \text{Attr}_1^1 = \{s_0, s_3, s_4\} \]
\[ \text{Attr}_0^2 = \{s_0, s_3, s_4, s_7\} \]
\[ \text{Attr}_0^3 = \{s_0, s_3, s_4, s_6, s_7\} \]
\[ \text{Attr}_0^4 = \{s_0, s_3, s_4, s_5, s_6, s_7\} \]
Example

\[
\begin{align*}
\text{Attr}_0^0 &= \{s_3, s_4\} \\
\text{Attr}_0^1 &= \{s_0, s_3, s_4\} \\
\text{Attr}_0^2 &= \{s_0, s_3, s_4, s_7\} \\
\text{Attr}_0^3 &= \{s_0, s_3, s_4, s_6, s_7\} \\
\text{Attr}_0^4 &= \{s_0, s_3, s_4, s_5, s_6, s_7\}
\end{align*}
\]
Computing the Attractor

Construction of $\text{Attr}_0^i(F)$:

$$\text{Attr}_0^0(F) = F$$
$$\text{Attr}_0^{i+1}(F) = \text{Attr}_0^i(F) \cup \text{ForceNext}_0(\text{Attr}_0^i(F))$$

Then $\text{Attr}_0^0(F) \subseteq \text{Attr}_0^1(F) \subseteq \text{Attr}_0^2(F) \subseteq \ldots$ and since $S$ is finite, there exists $k \leq |S|$ s.t. $\text{Attr}_0^k(F) = \text{Attr}_0^{k+1}(F)$.

The $0$-Attractor is defined as:

$$\text{Attr}_0(F) := \bigcup_{i=0}^{k} \text{Attr}_0^i(F)$$
Computing the Attractor

Construction of $\text{Attr}^i_0(F)$:

\[
\begin{align*}
\text{Attr}^0_0(F) &= F \\
\text{Attr}^{i+1}_0(F) &= \text{Attr}^i_0(F) \cup \text{ForceNext}_0(\text{Attr}^i_0(F))
\end{align*}
\]

Then $\text{Attr}^0_0(F) \subseteq \text{Attr}^1_0(F) \subseteq \text{Attr}^2_0(F) \subseteq \ldots$ and since $S$ is finite, there exists $k \leq |S|$ s.t. $\text{Attr}^k_0(F) = \text{Attr}^{k+1}_0(F)$.

The 0-Attractor is defined as:

\[
\text{Attr}_0(F) := \bigcup_{i=0}^{k} \text{Attr}^i_0(F)
\]

Claim: $W_0 = \text{Attr}_0(F)$ and $W_1 = S \setminus \text{Attr}_0(F)$
Duality Between Players

Assume we have a partition of the state space $S = P_0 \cup P_1$ (i.e., $P_0 \cap P_1 = \emptyset$) and we want to prove $W_0 = P_0$ and $W_1 = P_1$.

We want to prove $P_0 \supseteq W_0$, $P_0 \subseteq W_0$, $P_1 \supseteq W_1$, and $P_1 \subseteq W_1$.

Since we know that $W_0 \cap W_1 = \emptyset$ holds, it is sufficient to prove $P_0 \subseteq W_0$ and $P_1 \subseteq W_1$.

\[
\begin{align*}
  P_0 & \subseteq W_0 \\
  S \setminus P_0 & \supseteq S \setminus W_0 \\
  P_1 & \supseteq S \setminus W_0 \supseteq W_1 \\
  P_1 & \supseteq W_1 \\
  P_1 & \subseteq W_1 \\
  S \setminus P_1 & \supseteq S \setminus W_1 \\
  P_0 & \supseteq S \setminus W_1 \supseteq W_0 \\
  P_0 & \supseteq W_0
\end{align*}
\]
0-Attractor

To show $W_0 = \text{Attr}_0(F)$ and $W_1 = S \setminus \text{Attr}_0(F)$, we construct winning strategies for Player 0 and 1.
0-Attractor

To show $W_0 = \text{Attr}_0(F)$ and $W_1 = S \setminus \text{Attr}_0(F)$, we construct winning strategies for Player 0 and 1.

Proof.

$\text{Attr}_0(F) \subseteq W_0$

We prove for every $i$ and for every state $s \in \text{Attr}_0^i(F)$ that Player 0 has a positional winning strategy to reach $F$ in $\leq i$ steps.

- (Base) $s \in \text{Attr}_0^0(F) = F$
- (Induction) $s \in \text{Attr}_0^i(F)$

If $s \in \text{Attr}_0^i(F)$, then we apply induction hypothesis.

Otherwise $s \in \text{ForceNext}_0(\text{Attr}_0^i(F)) \setminus \text{Attr}_0^i(F)$ and Player 0 can force a visit to $\text{Attr}_0^i(F)$ in one step and from there she needs at move $i$ steps by induction hypothesis. So, $F$ is reached after at most $i + 1$ steps.
0-Attractor cont.

Proof cont.

\[ S \setminus \text{Attr}_0(F) \subseteq W_1 \]

Assume \( s \in S \setminus \text{Attr}_0(F) \), then \( s \notin \text{ForceNext}_0(\text{Attr}_0(F)) \) and we have two cases:

\( (a) \quad s \in S_0 \cap (S \setminus \text{Attr}_0(F)): \forall s' \in S. (s, s') \in E \rightarrow s' \notin \text{Attr}_0(F) \)

\( (b) \quad s \in S_1 \cap (S \setminus \text{Attr}_0(F)): \exists s' \in S. (s, s') \in E \land s' \notin \text{Attr}_0(F) \)

In \( S \setminus \text{Attr}_0(F) \) Player 1 can choose edges according to (b) leading again to \( S \setminus \text{Attr}_0(F) \) and by (a) Player 0 cannot escape from \( S \setminus \text{Attr}_0(F) \). So, \( F \) will be avoided forever.

\[ W_0 = \text{Attr}_0(F) \text{ and } W_1 = S \setminus \text{Attr}_0(F) \]
Example of Attractor Strategy
Example of Attractor Strategy
Example of Attractor Strategy
Example of Attractor Strategy
Example of Attractor Strategy
Safety Games

Given a safety game \((G, F)\) with \(G = (S, S_0, E)\), i.e.,

\[
\phi_S = \{\rho \in S^\omega \mid \text{Occ}(\rho) \subseteq F\},
\]

consider the reachability game \((G, S \setminus F)\), i.e.,

\[
\phi_R = \{\rho \in S^\omega \mid \text{Occ}(\rho) \cap (S \setminus F) \neq \emptyset\}.
\]

Then,

\[
S^\omega \setminus \phi_R = \{\rho \in S^\omega \mid \text{Occ}(\rho) \cap (S \setminus F) = \emptyset\} = \{\rho \in S^\omega \mid \text{Occ}(\rho) \subset F\}.
\]

Player 0 has a safety objective in \((G, F)\) \iff Player 1 has a reachability objective in \((G, S \setminus F)\).

So, \(W_0\) in the safety game \((G, F)\) corresponds to \(W_1\) in the reachability game \((G, S \setminus F)\).
Summary

We know how to solve reachability and safety games by positional winning strategies.

The strategies are

- Player 0: Decrease distance to $F$
- Player 1: Stay outside of $\text{Attr}_0(F)$

In LTL, $\Diamond F =$ reachability and $\Box F =$ safety.

Next, $\Box \Diamond F =$ Büchi and $\Diamond \Box F =$ Co-Büchi.
Hierarchy

4. Reactivity: Muller, Parity

3. Recurrence: Büchi  Persistence: co-Büchi

2. Obligation: Staiger-Wagner, Weak-Parity

1. Reachability  Safety
Büchi and co-Büchi Games
**Büchi Game**

Given a Büchi game \((G, F)\) over the game graph \(G = (S, S_0, E)\) with the set \(F \subseteq S\) of Büchi states, we aim to

- determine the winning regions of Player 0 and 1
- compute their respective winning strategies

Recall, Player 0 wins \(\rho\) iff she visits infinitely often states in \(F\), i.e.,
\[
\phi = \{\rho \in S^\omega \mid \inf(\rho) \cap F \neq \emptyset\}.
\]
Idea

Compute for $i \geq 1$ the set $\text{Recur}_0^i$ of accepting states $s \in F$ from which Player 0 can force at least $i$ revisits to $F$. Then, we will show that

$$F \supseteq \text{Recur}_0^1(F) \supseteq \text{Recur}_0^2(F) \supseteq \ldots$$

and we compute the winning region of Player 0 with

$$\text{Recur}_0(F) := \bigcap_{i \leq 1} \text{Recur}_0^i(F)$$

Since $F$ is finite, there exists $k$ such that $\text{Recur}_0(F) = \text{Recur}_0^k(F)$. 
One-Step Attractor

First, we define Recur$_0$ formally using a modified version of Attractor. We count revisits, so we need the set of states from which Player 0 can force a revisit to $F$, i.e., state from which she can force a visit in $\geq 1$ steps.

We define a slightly modified attractor:

\[
A^0_0 = \emptyset \\
A^{i+1}_0 = A^i_0 \cup \text{ForceNext}_0(A^i_0 \cup F)
\]

\[
\text{Attr}_0^+(F) = \bigcup_{i \geq 0} A^i_0
\]
Visit versus Revisit

\[ \text{Attr}_0(F) \]

\[ \text{Attr}_{0}^{+}(F') \]
Recurrence Set

We define

\[
\text{Recur}_0^0(F) := F \\
\text{Recur}_0^{i+1}(F) := F \cap \text{Attr}_0^+(\text{Recur}_0^i(F)) \\
\text{Recur}_0(F) := \bigcap_{i \geq 0} \text{Recur}_0^i(F)
\]

We show that there exists \( k \) such that \( \text{Recur}_0(F) := \bigcap_{i \geq 0} \text{Recur}_0^i(F) \) by proving \( \text{Recur}_0^{i+1}(F) \subseteq \text{Recur}_0^i(F) \) for all \( i \geq 0 \).

Proof.

\[ i = 0: \quad F \cap \text{Attr}_0^+(F) \subseteq F \]

\[ i \rightarrow i + 1: \]

\[
\text{Recur}_0^{i+1}(F) = F \cap \text{Attr}_0^+(\text{Recur}_0^i(F)) \subseteq F \cap \text{Attr}_0^+(\text{Recur}_0^{i-1}(F)) = \text{Recur}_0^i(F) \text{ since (i) } \text{Recur}_0^i(F) \subseteq \text{Recur}_0^{i-1}(F) \text{ by ind. hyp. and (ii) } \text{Attr}_0^+ \text{ is monotone.}
\]
Recurrence Set cont.
Recurrence Set cont.

We show that all states in \( \text{Attr}_0(\text{Recur}_0(F)) \) are winning for Player 0, i.e., \( \text{Attr}_0(\text{Recur}_0(F)) \subseteq W_0 \). We construct a memoryless winning strategy for Player 0 for all states in \( \text{Attr}_0(\text{Recur}_0(F)) \).

Proof.

We know that there exists \( k \) such that
\[
\text{Recur}_0^{k+1}(F) = \text{Recur}_0^k(F) = F \cap \text{Attr}_0^+(\text{Recur}_0^k(F)).
\]
So,

\[
\text{(i)} \quad \text{for } s \in \text{Recur}_0^k(F) \cap S_0 \text{ Player 0 can choose an edge back to } \text{Attr}_0^+(\text{Recur}_0^k(F)) \text{ and }
\]
\[
\text{(ii)} \quad \text{for } s \in \text{Recur}_0^k(F) \cap S_1 \text{ all edges lead back to } \text{Attr}_0^+(\text{Recur}_0^k(F)).
\]

For all states in \( \text{Attr}_0(\text{Recur}_0(F)) \setminus \text{Recur}_0(F) \), Player 0 can follow the attractor strategy to reach \( \text{Recur}_0(F) \).
Recurrence Set cont.

We show $S \setminus \text{Attr}_0(\text{Recur}_0(F)) \subseteq W_1$.

Proof.

We know that there exists $k$ such that $\text{Recur}_0(F) = \text{Recur}_{0}^{k}(F)$, i.e., $S \setminus \text{Attr}_0(\text{Recur}_0(F)) = S \setminus \text{Attr}_0(\text{Recur}_{0}^{k}(F))$.

Show: Player 1 can force $\leq i$ visits to $F$ from $s \notin \text{Attr}_0(\text{Recur}_{0}^{i}(F))$

$i = 0$: $s \notin \text{Attr}_0(F)$, so Player 1 can avoid visiting $F$ at all.

$i \to i + 1$: $s \notin \text{Attr}_0(\text{Recur}_{0}^{i+1}(F))$.

- $s \notin \text{Attr}_0(\text{Recur}_{0}^{i}(F))$, Player 1 plays according to ind. hypothese

- Otherwise, $s \in \text{Attr}_0(\text{Recur}_{0}^{i}(F)) \setminus \text{Attr}_0(\text{Recur}_{0}^{i+1}(F))$ and Player 1 can avoid $\text{Attr}_0(\text{Recur}_{0}^{i+1}(F))$. 
Büchi games

We have shown that Player 0 has a (memoryless) winning strategy from every state in Attr_0(Recur_0(F)), so Attr_0(Recur_0(F)) ⊆ W_0. And, Player 1 has a (memoryless) winning strategy from every state in S \ Attr_0(Recur_0(F)), so S \ Attr_0(Recur_0(F)) ⊆ W_1. This implies the following theorem.

Theorem

Given a Büchi game ((S, S_0, E), F), the winning regions W_0 and W_1 are computable and form a partition, i.e., W_0 ∪ W_1 = S. Both players have memoryless winning strategies.
Co-Büchi Games

Given a Co-Büchi Game $((S, S_0, E), F)$, i.e.,

$$
\phi_C = \{ \rho \in S^\omega \mid \text{Inf}(\rho) \subseteq F \}
$$

consider the Büchi Game $((S, S_0, E), S \setminus F)$, i.e,

$$
\phi_B = \{ \rho \in S^\omega \mid \text{Inf}(\rho) \cap (S \setminus F) \neq \emptyset \}.
$$

Then, $S^\omega \setminus \phi_B = \{ \rho \in S^\omega \mid \text{Inf}(\rho) \cap (S \setminus F) = \emptyset \}$

$$
= \{ \rho \in S^\omega \mid \text{Inf}(\rho) \subseteq F \}.
$$

Player 0 has a co-Büchi objective in $(G, F)$ $\iff$

Player 1 has a Büchi objective in $(G, S \setminus F)$.

So, $W_0$ in the co-Büchi game $(G, F)$ corresponds to $W_1$ in the Büchi game $(G, S \setminus F)$. 

Summary

We know how to solve Büchi and Co-Büchi games by positional winning strategies.

In LTL,

- $\Diamond F = \text{reachability}$
- $\Box F = \text{safety}$
- $\Box \Diamond F = \text{Büchi}$
- $\Diamond \Box F = \text{Co-Büchi}$
**Exercise**

2. Consider the game graph shown below and the following winning conditions:

   (a) \( \text{Occ}(\rho) \cap \{1\} \neq \emptyset \) and 
   (b) \( \text{Occ}(\rho) \subseteq \{1, 2, 3, 4, 5, 6\} \) and 
   (c) \( \text{Inf}(\rho) \cap \{4, 5\} \neq \emptyset \).

Compute the winning regions and corresponding winning strategies showing the intermediate steps (i.e., the Attractor and Recurrence sets) of the computation.