Automata on Infinite Trees
Büchi Automata on Infinite Trees
Definition

A Büchi tree automaton over $\Sigma$ is $A = \langle S, I, T, F \rangle$, where:

- $S$ is a finite set of states,
- $I \subseteq S$ is a set of initial states,
- $T : S \times \Sigma \rightarrow 2^{S \times S}$ is the transition relation,
- $F \subseteq S$ is the set of final states.
**Runs**

A *run* of $A$ over a tree $t : \{0, 1\}^* \to \Sigma$ is a mapping $\pi : \{0, 1\}^* \to S$ such that, for each position $p \in \{0, 1\}^*$, where $q = \pi(p)$, we have:

- if $p = \epsilon$ then $q \in I$, and

- if $q_i = \pi(p_i)$, $i = 0, 1$ then $\langle q_0, q_1 \rangle \in T(q, t(p))$.

If $\pi$ is a run of $A$ and $\sigma$ is a path in $t$, let $\pi|\sigma$ denote the path in $\pi$ corresponding to $\sigma$.

A run $\pi$ is said to be *accepting*, if and only if for every path $\sigma$ in $t$ we have:

$$\inf(\pi|\sigma) \cap F \neq \emptyset$$
Closure Properties

For every Büchi automaton $A$ there exists a complete Büchi automaton $A'$ such that $\mathcal{L}(A) = \mathcal{L}(A')$.

**Theorem 1** The class of Büchi-recognizable tree languages is closed under union, intersection and projection.

Let $A_i = \langle S_i, I_i, T_i, F_i \rangle$, $i = 1, 2$, where $S_1 \cap S_2 = \emptyset$.

Let $A_\cup = \langle S_1 \cup S_2, I_1 \cup I_2, T_1 \cup T_2, F_1 \cup F_2 \rangle$. 
Closure Properties

Let $A_\cap = \langle S, I, T, F \rangle$ where:

- $S = S_1 \times S_2 \times \{0, 1, 2\}$
- $I = I_1 \times I_2 \times \{1\}$
- for any $s, s_1, s_2 \in S_1, s', s'_1, s'_2 \in S_2$, $a, b \in \{0, 1, 2\}$:
  \[
  \langle (s_1, s'_1, b), (s_2, s'_2, b) \rangle \in T((s, s', a), \sigma)
  \]
  
  iff $\langle s_1, s_2 \rangle \in T(s, \sigma)$, $\langle s'_1, s'_2 \rangle \in T(s', \sigma)$ and:

  1. if $a = 0$ or ($a = 1$ and $s \notin F_1$), then $b = 1$
  2. if ($a = 1$ and $s \in F_1$) or ($a = 2$ and $s' \notin F_2$), then $b = 2$
  3. if $a = 2$ and $s' \in F_2$, then $b = 0$
- $F = S \times S \times \{0\}$
Emptiness of Büchi Tree Automata

Let $A = \langle S, I, T, F \rangle$ be a Büchi tree automaton where $F = \{s_1, \ldots, s_m\}$, and $\pi : \{0, 1\}^* \to S$ be an accepting run of $A$ on the tree $t \in \mathcal{T}(\Sigma)$.

For any $s \in S$, and any $u \in \{0, 1\}^*$ such that $\pi(u) = s$, let

$$d^\pi_u = \{w \in u \cdot \{0, 1\}^* \mid \forall v. u < v < w \Rightarrow \pi(v) \notin F\}$$

By König’s lemma, $d^\pi_u$ is finite for any $u \in \{0, 1\}^*$.

Let $t^\pi_s$ be the restriction of $t$ to $d^\pi_u$. Let

$$T_s = \{t^\pi_s \mid \pi \text{ is an accepting run of } A \text{ on } t\}$$
Emptiness of Büchi Tree Automata

If $\vec{s} = \langle s_1, \ldots, s_m \rangle$ are the final states of $A$:

$$\mathcal{L}(A) = \bigcup_{s_0 \in I} T_{s_0} \cdot \vec{s} \langle T_{s_1}, \ldots, T_{s_m} \rangle_{\omega} \vec{s}$$

Conversely, the expression above denotes a Büchi-recognizable tree language.

Let $A = \langle S, I, T, F \rangle$ be a Büchi tree automaton. For each $s \in S$ let $T_s$ be the recognizable tree language defined above. Eliminate from $S$ (and $T$) all states $s$ such that $T_s = \emptyset$, and let $S'$ be the resulting set of states.

We claim that $\mathcal{L}(A) \neq \emptyset \iff S' \cap I \neq \emptyset$. 
The Complement Problem

Let $\Sigma = \{a, b\}$, $\mathcal{T}_0 = \{t \in \mathcal{T}^\omega(\Sigma) \mid \text{some path in } t \text{ has infinitely many } a\text{'s}\}$

$\mathcal{T}_0$ is Büchi recognizable.

Let $A = \langle \{s_0, s_1, s_a, s_b\}, \{s_0\}, T, \{s_1, s_a\} \rangle$, where $T$ is defined by:

- $a(s_0, a, b) \rightarrow \{\langle s_1, s_a \rangle, \langle s_a, s_1 \rangle\}$
- $b(s_0, a, b) \rightarrow \{\langle s_1, s_b \rangle, \langle s_b, s_1 \rangle\}$
- $a(s_1) \rightarrow \{\langle s_1, s_1 \rangle\}$
- $b(s_1) \rightarrow \{\langle s_1, s_1 \rangle\}$
The Complement Problem

Let $\mathcal{T}_1 = \mathcal{T}^\omega(\Sigma) \setminus \mathcal{T}_0 = \{t \in \mathcal{T}^\omega(\Sigma) \mid \text{all paths in } t \text{ have finitely many } a\text{'s}\}$. We show that $\mathcal{T}_1$ cannot be recognized by a Büchi tree automaton.

**Exercise 1** $I = \{s_0, s_1\}$, $F = \{s_1\}$ and

\[
\begin{align*}
a(s_0) &\rightarrow \langle s_0, s_0 \rangle \\
&\langle s_0, s_1 \rangle \\
&\langle s_1, s_0 \rangle \\
&\langle s_1, s_1 \rangle \\
b(s_0) &\rightarrow \langle s_0, s_0 \rangle \\
&\langle s_0, s_1 \rangle \\
&\langle s_1, s_0 \rangle \\
&\langle s_1, s_1 \rangle \\
b(s_1) &\rightarrow \langle s_1, s_1 \rangle
\end{align*}
\]
The Complement Problem

Let \( T_n : \{0, 1\}^* \rightarrow \Sigma \) be the language of trees:

\[
t_n(p) = \begin{cases} 
a & \text{if } p \in \{\epsilon, 1^{m_1}0, 1^{m_1}01^{m_2}0, \ldots, 1^{m_1}01^{m_2}0 \ldots 1^{m_n}0 \mid m_1, \ldots, m_n \in \mathbb{N}\} 
b & \text{otherwise} \end{cases}
\]

Obviously, \( T_n \subset T_1 \), for all \( n \in \mathbb{N} \).

Suppose there exists a Büchi automaton \( A = \langle S, I, T, F \rangle \) with \( k \) states, s.t. \( \mathcal{L}(A) = T_1 \). Let \( \pi \) be the accepting run of \( A \) over \( t_{k+1} \). Then there exist:

- \( m_1 > 0 \) such that \( \pi(1^{m_1}) = s_1 \in F \)
- \( m_2 > 0 \) such that \( \pi(1^{m_1}01^{m_2}) = s_2 \in F \)
- \( \ldots \)

There exists a path \( \sigma \) in \( t_m \) and \( u < v < w < \sigma \), such that \( \pi(u) = \pi(w) = s \in F \) and \( t_m(v) = a \). Then \( \pi = r_1 \cdot s \cdot r_2 \cdot s \cdot r_3 \), and \( r_1 \cdot s \cdot r_2^\omega s \) is an accepting run on \( q_1 \cdot q_2^\omega \), which contains a path with infinitely many \( a \).
Muller Automata on Infinite Trees
**Definition**

A Muller tree automaton $\Sigma$ is $A = \langle S, I, T, F \rangle$, where:

- $S$ is a finite set of *states*,
- $I \subseteq S$ is a set of *initial states*,
- $T : S \times \Sigma \rightarrow 2^{S \times S}$ is the *transition function*,
- $F \subseteq 2^S$, is the set of *accepting sets*.

A run $\pi$ of $A$ over $t$ is said to be *accepting*, iff for every path $\sigma$ in $t$:

$$\inf(\pi|_\sigma) \in \mathcal{F}$$
Closure Properties

The class of Muller-recognizable tree languages is closed under union and intersection.

For **union**, the proof is exactly as in the case of Büchi automata. For $A_\cup$, the set of accepting sets is the union of the sets $\mathcal{F}_i, i = 1, 2$.

For **intersection**, let $A_\cap = \langle S_1 \times S_2, I_1 \times I_2, T, \mathcal{F} \rangle$, where:

- $\langle (s_1, s'_1), (s_2, s'_2) \rangle \in T((s, s'), \sigma)$ iff $\langle s_1, s_2 \rangle \in T(s, \sigma)$ and $\langle s'_1, s'_2 \rangle \in T(s', \sigma)$, and

- $\mathcal{F} = \{ G \in S_1 \times S_2 \mid pr_1(G) \in \mathcal{F}_1 \text{ and } pr_2(G) \in \mathcal{F}_2 \}$, where:
  - $pr_1(G) = \{ s \in S_1 \mid \exists s'. (s, s') \in G \}$, and
  - $pr_2(G) = \{ s \in S_2 \mid \exists s'. (s', s) \in G \}$. 

Rabin Automata on Infinite Trees
**Definition**

A **Rabin** tree automaton \( \Sigma \) is \( A = \langle S, I, T, \Omega \rangle \), where:

- \( S \) is a finite set of **states**, 
- \( I \subseteq S \) is a set of **initial states**, 
- \( T : S \times \Sigma \to 2^{S \times S} \) is the **transition function**, 
- \( \Omega = \{ \langle N_1, P_1 \rangle, \ldots, \langle P_n, N_n \rangle \} \) is the set of **accepting pairs**.

A run \( \pi \) of \( A \) over \( t \) is said to be **accepting**, if and only if for every path \( \sigma \) in \( t \) there exists a pair \( \langle N_i, P_i \rangle \in \Omega \) such that:

\[
\inf(\pi|_{\sigma}) \cap N_i = \emptyset \text{ and } \inf(\pi|_{\sigma}) \cap P_i \neq \emptyset
\]
Büchi, Muller and Rabin

For every Büchi tree automaton $A$ there exists a Muller tree automaton $B$, such that $\mathcal{L}(A) = \mathcal{L}(B)$, but not viceversa.

For every Muller tree automaton $A$ there exists a Rabin tree automaton $B$, such that $\mathcal{L}(A) = \mathcal{L}(B)$, and viceversa.
From Büchi to Muller

For each (nondeterministic) Büchi automaton $A$ there exists a (nondeterministic) Muller automaton $B$ such that $\mathcal{L}(A) = \mathcal{L}(B)$

Let $A = \langle S, I, T, F \rangle$ be a Büchi automaton.

Define $B = \langle S, I, T, \{G \in 2^S \mid G \cap F \neq \emptyset\} \rangle$

Allowing Muller automata to be nondeterministic is essential here.
From Rabin to Muller

Given a Rabin automaton $A = \langle S, I, T, \Omega \rangle$, such that

$$\Omega = \{ \langle N_1, P_1 \rangle, \ldots, \langle N_k, P_k \rangle \}$$

let $B = \langle S, I, T, \mathcal{F} \rangle$ be the Muller automaton, where

$$\mathcal{F} = \{ F \subseteq S \mid F \cap N_i = \emptyset \text{ and } F \cap P_i \neq \emptyset \text{ for some } 1 \leq i \leq k \}$$
From Muller to Rabin

Given a Muller automaton $A = \langle S, I, T, \mathcal{F} \rangle$, there exists a Rabin automaton $B$ such that $\mathcal{L}(A) = \mathcal{L}(B)$.

Let $\mathcal{F} = \{Q_1, \ldots, Q_k\}$

Let $B = \langle S', I', T', \Omega' \rangle$ where:

- $S' = 2^{Q_1} \times \ldots \times 2^{Q_k} \times S$
- $I' = \{\langle \emptyset, \ldots, \emptyset, s_0 \rangle \mid s_0 \in I\}$
From Muller to Rabin

- $T'(\langle S_1, \ldots, S_k, s \rangle, a) = (\langle S'_1, \ldots, S'_k, s' \rangle, \langle S''_1, \ldots, S''_k, s'' \rangle)$ where:
  - $T(s, a) = (s', s'')$
  - for all $1 \leq i \leq k$:
    $$S'_i = S''_i = \begin{cases} \emptyset & \text{, if } S_i \cup \{s\} = Q_i \\ (S_i \cup \{s\}) \cap Q_i & \text{, otherwise} \end{cases}$$

- $P_i = \{\langle S_1, \ldots, S_i, \ldots, S_k, s \rangle \mid S_i = Q_i\}, 1 \leq i \leq k$

- $N_i = \{\langle S_1, \ldots, S_i, \ldots, S_k, s \rangle \mid s \not\in Q_i\}, 1 \leq i \leq k$
The Rabin Complementation Theorem

Theorem 2 (Rabin ’69) The class of Rabin-recognizable tree languages is closed under complement.

The class of Rabin-recognizable tree languages is closed under union and intersection, because Muller-recognizable languages are.
Emptiness of Rabin Automata

Given an alphabet $\Sigma$, an infinite tree $t \in T^\omega(\Sigma)$ is said to be *regular* if there are only finitely many distinct subtrees $t_u$ of $t$, where $u \in \{0, 1\}^\ast$.

**Example 1** The infinite binary tree $f(g(f(\ldots), f(\ldots)), g(f(\ldots), f(\ldots)))$ is regular.  

**Theorem 3 (Rabin ‘72)**

1. Any non-empty Rabin-recognizable set of trees contains a regular tree.
2. The emptiness problem for Rabin tree automata is decidable.
Reduction to empty alphabet

Let $A = \langle S, I, T, \Omega \rangle$ be a Rabin tree automaton over $\Sigma$, such that $L(A) \neq \emptyset$, where $\Omega = \{\langle N_1, P_1 \rangle, \ldots, \langle N_n, P_n \rangle\}$.

Let $A' = \langle S \times \Sigma, I \times \Sigma, T', \Omega' \rangle$, where:

- $\langle (s_1, \sigma_1), (s_2, \sigma_2) \rangle \in T'((s, \sigma))$ iff $\langle s_1, s_2 \rangle \in T(s, \sigma)$, and $\sigma_1, \sigma_2 \in \Sigma$.
- $\Omega' = \{\langle N_1 \times \Sigma, P_1 \times \Sigma \rangle, \ldots, \langle N_n \times \Sigma, P_n \times \Sigma \rangle\}$.

The accepting runs of $A'$ are pairs $(\pi, t)$, where $t \in L(A)$, and $\pi$ is a accepting run of $A$ on $t$. 
**Regular accepting runs**

For any Rabin tree automaton $A$, there exists a Rabin tree automaton $A'$ with one initial state such that $\mathcal{L}(A) = \mathcal{L}(A')$.

Consider a Rabin tree automaton $A = \langle S, s_0, T, \Omega \rangle$ over the empty alphabet, and let $\pi$ be an accepting run of $A$.

**Claim 1** If $A$ has an accepting run, $A$ has also a regular accepting run.

A state $s \in S$ is said to be *live* if $s \neq s_0$ and $\langle s_1, s_2 \rangle \in T(s)$ for some $s_1, s_2 \in S$, where either $s_1 \neq s$ or $s_2 \neq s$.

By induction on $n = \text{the number of live states in } A$. 
Regular accepting runs

Base case $n = 0$: $\pi(\epsilon) = s_0$ and $\pi(p) = s$, for all $p \in \text{dom}(\pi)$, and $s \in S$ is non-live.

Inductive step $n > 0$:

Case 1 If some live state in $A$ is missing on $\pi$, apply the induction hypothesis.

Case 2 All live states of $A$ appear on $\pi$, and there is a position $u \in \{0, 1\}^*$ such that $\pi(u) = s$ is live, but some live state $s'$ does not appear in $\pi_u$.

Let $\pi_1 = \pi \setminus \pi_u$ and $\pi_2 = \pi_u$. Both $\pi_1$ and $\pi_2$ are runs of automata with $n - 1$ live states, hence there exists accepting regular runs $\pi'_1$ and $\pi'_2$ of these automata. The desired run is $\pi'_1 \cdot_s \pi'_2$. 
Regular accepting runs

Case 3 All live states appear in any subtree of $\pi$. Let $\sigma$ be a path in $\pi$ consisting of all the live states appearing again and again, and only of the live states, with the exception of $\pi(\epsilon)$. Q: Why does $\sigma$ exist?

There exists $\langle N, P \rangle \in \Omega$, such that $\inf(\sigma) \cap N = \emptyset$ and $\inf(\sigma) \cap P \neq \emptyset$. Then $N$ contains only non-live states.

Let $s \in \inf(\sigma) \cap P$ and $u, v$ be the $1^{st}$ and $2^{nd}$ positions such that $\sigma(u) = \sigma(v) = s$.

Let $\pi_1 = \pi \setminus \pi_u$ and $\pi_2 = \pi_u \setminus \pi_v$. Both $\pi_1$ and $\pi_2$ are accepting runs of automata with $n - 1$ live states, hence there exists accepting regular runs $\pi'_1$ and $\pi'_2$ of these automata. The desired run is $\pi'_1 \cdot s \pi'_2 \omega s$. 
The Emptiness Problem

Let $A$ be an input-free Rabin tree automaton with $n$ live states.

We derive $A_{n-1}, A_{n-2}, \ldots, A_0$ from $A$, having $n-1, n-2, \ldots, 0$ live states.

If $A$ has a accepting run, then it it has a regular run, composed of runs of $A_{n-1}, A_{n-2}, \ldots, A_0$.

So it is enough to check emptiness of $A_{n-1}, A_{n-2}, \ldots, A_0$. 
Rabin Automata, SkS and SωS
Defining infinite paths

We say that a set of positions $X$ is linear iff the following holds:

$$linear(X) : (\forall x, y . X(x) \land X(y) \rightarrow x \leq y \lor y \leq x)$$

$X$ is a path iff:

$$path(X) : linear(X) \land \forall Y . linear(Y) \land X \subseteq Y \rightarrow X = Y$$
From Automata to Formulae

Let $A = \langle S, I, T, \Omega \rangle$ be a Rabin tree automaton, where $S = \{s_1, \ldots, s_p\}$.

Let $\vec{Y} = \{Y_1, \ldots, Y_p\}$ be set variables.

If $X$ denotes a path, state $i$ appears infinitely often in $X$ iff:

$$\inf_i(X) : \forall x . X(x) \rightarrow \exists y . x \leq y \wedge X(y) \wedge Y_i(y)$$

The formula expressing the accepting condition is:

$$\Phi_\Omega(\vec{Y}) : \forall X . \text{path}(X) \rightarrow \bigvee_{\langle N,P \rangle \in \Omega} \left( \bigwedge_{s_i \in N} \neg \inf_i(X) \wedge \bigvee_{s_i \in P} \inf_i(X) \right)$$
Decidability of S2S

**Theorem 4** Given an alphabet $\Sigma$, a tree language $L \subseteq T^\omega(\Sigma)$ is definable in S2S iff it is recognizable.

**Corollary 1** The SAT problem for S2S is decidable.
Obtaining Decidability Results by Reduction

Suppose we have a logic $\mathcal{L}$ interpreted over the domain $\mathcal{D}$, such that the following problem is decidable:

\[
\text{for each formula } \varphi \text{ of } \mathcal{L} \text{ there exists } m \in \mathcal{D} \text{ such that } m \models \varphi
\]

Then we can prove the same thing for another logic $\mathcal{L}'$ interpreted over $\mathcal{D}'$ iff there exists functions $\Delta : \mathcal{D}' \rightarrow \mathcal{D}$ and $\Lambda : \mathcal{L}' \rightarrow \mathcal{L}$ such that for all $m' \in \mathcal{D}'$ and $\varphi' \in \mathcal{L}$ we have:

\[
m' \models \varphi' \iff \Delta(m') \models \Lambda(\varphi')
\]
Decidability of $S_\omega S$

Every tree $t : \mathbb{N}^* \rightarrow \Sigma$ can be encoded as $t' : \{0, 1\}^* \rightarrow \Sigma$.

Let $D = \{\epsilon\} \cup \{1^{n_1+1}01^{n_2+1}0 \ldots 1^{n_k+1}0 \mid k \geq 1, \ n_i \in \mathbb{N}, \ 1 \leq i \leq k\}$.

Embedding the domain of $S_\omega S$ into $S2S$:

$$D(x) \quad : \quad \exists z \forall y . \ z \leq y \land x = z \lor \forall y . \ s_0(y) \leq x \rightarrow \exists y' . \ y = s_1(y')$$
Decidability of $S\omega S$

If $p = 1^{n_1+1}01^{n_2+1}0 \ldots 1^{n_k+1}0$, let

$$f_i(p) = p \cdot 1^{i+1}0 = 1^{n_1+1}01^{n_2+1}0 \ldots 1^{n_k+1}01^{i+1}0$$

$$x \leq_D y : D(x) \land D(y) \land x \leq y$$

Define the relation $x \leq^3_D y$ iff $x \in D$ and $y = x \cdot 1^{n+1}0$, for some $n \in \mathbb{N}$:

$$x \leq^3_D y : \exists z . y = s_0(z) \land \forall z' . x \leq z \land z' < z \rightarrow s_1(z') \leq y$$

Define $f_0, f_1, f_2, \ldots$ by induction:

$$f_0(x) = y : D(x) \land D(y) \land \exists z . y = s_0(z) \land z = s_1(x)$$

$$f_{i+1}(x) = y : D(x) \land D(y) \land x \leq^3_D y \land \forall z . x \leq^3_D z \land$$

$$\land_{0 \leq k \leq i} z \neq f_k(x) \rightarrow y \leq_D z$$