Automata on Infinite Words

Definition of Büchi Automata

Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet.

A *non-deterministic Büchi automaton* (NBA) over Σ is a tuple $A = \langle S, I, T, F \rangle$, where:

- S is a finite set of *states*,
- $I \subseteq S$ is a set of *initial states*,
- $T \subseteq S \times \Sigma \times S$ is a *transition relation*,
- $F \subseteq S$ is a set of *final states*.

Acceptance Condition

A *run* of a Büchi automaton is defined over an infinite word $w : \alpha_1\alpha_2\ldots$ as an infinite sequence of states $\pi \ : \ s_0 s_1 s_2 \ldots$ such that:

- $s_0 \in I$ and
- $(s_i, \alpha_{i+1}, s_{i+1}) \in T$, for all $i \in \mathbb{N}$.

$$\boxed{\inf(\pi) = \{s \mid s \text{ appears infinitely often on } \pi\}}$$

Run π of A is said to be *accepting* iff $\inf(\pi) \cap F \neq \emptyset$.

The language of A, denoted $\mathcal{L}(A)$, is the set of all words accepted by A.

A language $L \subseteq \Sigma^\omega$ is *ω-recognizable* if there exists a Büchi automaton A such that $L = \mathcal{L}(A)$.

Examples

Let $\Sigma = \{0, 1\}$. Define Büchi automata for the following languages:

1. $L = \{\alpha \in \Sigma^\omega \mid 0 \text{ occurs in } \alpha \text{ exactly once}\}$
2. $L = \{\alpha \in \Sigma^\omega \mid \text{after each } 0 \text{ in } \alpha \text{ there is } 1\}$
3. $L = \{\alpha \in \Sigma^\omega \mid \alpha \text{ contains finitely many 1's}\}$
4. $L = (01)^*\Sigma^\omega$
5. $L = \{\alpha \in \Sigma^\omega \mid 0 \text{ occurs on all even positions in } \alpha\}$

Büchi Characterization Theorem

Lemma 1 *If* $L \subseteq \Sigma^*$ *is a recognizable language, there exists a DFA* $A = \langle S, \{s_0\}, T, F\rangle$ *such that* s_0 *has no incoming transitions and* $L = \mathcal{L}(A)$.

Given $W \subseteq \Sigma^*$, define $W^\omega = \{w_0 w_1 \ldots \mid w_i \in W, i \geq 0\}$

Lemma 2 *Let* $W, V \subseteq \Sigma^*$ *be recognizable languages. Then the languages* W^ω *and* $V \cdot W^\omega$ *are* ω*-recognizable.*

Büchi Characterization Theorem

Let $A = \langle S, I, T, F \rangle$ be a Büchi automaton and $s, s' \in S$ be two states.

Let $W_{s,s'} = \{w \in \Sigma^* \mid s \xrightarrow{w} s'\}$.

The language $W_{s,s'} \subseteq \Sigma^*$ is recognizable, for any $s, s' \in S$.

Theorem 1 *An ω-langage $L \subseteq \Sigma^\omega$ is ω-recognizable iff L is a finite union of ω-languages $V \cdot W^\omega$, where $V, W \subseteq \Sigma^*$ are recognizable languages.*

Proof idea: $L = \bigcup_{s \in I, s' \in F} W_{s,s'} \cdot W^\omega_{s',s'}$

Corollary 1 *Any non-empty Büchi-recognizable language contains an ultimately periodic word of the form $uvvv\ldots$.*

The Emptiness Problem

Theorem 2 *Given a Büchi automaton A, $\mathcal{L}(A) \neq \emptyset$ iff there exist $u, v \in \Sigma^*$, $|u|, |v| \leq \|A\|$, such that $uv^\omega \in \mathcal{L}(A)$.*

In practical terms, A is non-empty iff there exists a state s which is reachable both from an initial state and from itself.

Closure Properties

Closure under union and projection are like in the finite automata case.

Intersection is a bit special.

Complementation of non-deterministic Büchi automata is a complex result.

Deterministic Büchi automata are not closed under complement.

Closure under Intersection

Let $A_1 = \langle S_1, I_1, T_1, F_1 \rangle$ and $A_2 = \langle S_2, I_2, T_2, F_2 \rangle$

Build $A_\cap = \langle S, I, T, F \rangle$:

- $S = S_1 \times S_2 \times \{1, 2, 3\}$,
- $I = I_1 \times I_2 \times \{1\}$,
- the definition of T is the following:
  - $((s_1, s_2, 1), a, (s'_1, s'_2, 1)) \in T$ iff $(s_i, a, s'_i) \in T_i$, $i = 1, 2$ and $s_1 \notin F_1$
  - $((s_1, s_2, 1), a, (s'_1, s'_2, 2)) \in T$ iff $(s_i, a, s'_i) \in T_i$, $i = 1, 2$ and $s_1 \in F_1$
  - $((s_1, s_2, 2), a, (s'_1, s'_2, 2)) \in T$ iff $(s_i, a, s'_i) \in T_i$, $i = 1, 2$ and $s'_1 \notin F_2$
  - $((s_1, s_2, 2), a, (s'_1, s'_2, 3)) \in T$ iff $(s_i, a, s'_i) \in T_i$, $i = 1, 2$ and $s'_1 \in F_2$
  - $((s_1, s_2, 3), a, (s'_1, s'_2, 1)) \in T$ iff $(s_i, a, s'_i) \in T_i$, $i = 1, 2$
- $F = S_1 \times S_2 \times \{3\}$

Deterministic Büchi Automata

$\boxed{\omega\text{-languages recognized by NBA} \supsetneq \omega\text{-languages recognized by DBA}}$

Q: Why classical subset construction does not work for Büchi automata?

Let $A = \langle S, I, T, F \rangle$ and $A_d = \langle 2^S, \{I\}, T_d, \{Q \mid Q \cap F \neq \emptyset\} \rangle$.

Let $u_0 u_1 u_2 \ldots \in \mathcal{L}(A)$ be an infinite word. In A_d this gives:

$$I \xrightarrow{u_0} Q_1 \xrightarrow{u_1} Q_2 \xrightarrow{u_2} \ldots$$

where each $Q_i \cap F$. However this does not necessarily correspond to an accepting path in A!

Deterministic Büchi Automata

Let $W \subseteq \Sigma^*$. Define $\overrightarrow{W} = \{\alpha \in \Sigma^\omega \mid \alpha(0,n) \in W \text{ for infinitely many } n\}$

Theorem 3 *A language $L \subseteq \Sigma^\omega$ is recognizable by a deterministic Büchi automaton iff there exists a recognizable language $W \subseteq \Sigma^*$ such that $L = \overrightarrow{W}$.*

If $L = \mathcal{L}(A)$ then $W = \mathcal{L}(A')$ where A' is the DFA with the same definition as A, and with the finite acceptance condition.

Deterministic Büchi Automata

Theorem 4 *There exists an ω-recognizable language that can be recognized by no deterministic Büchi automaton.*

$\Sigma = \{a, b\}$ and $L = \{\alpha \in \Sigma^\omega \mid \#_a(\alpha) < \infty\} = \Sigma^* b^\omega$.

Suppose $L = \overrightarrow{W}$ for some $W \subseteq \Sigma^*$.

$b^\omega \in L \Rightarrow b^{n_1} \in W$

$b^{n_1} a b^\omega \in L \Rightarrow b^{n_1} a b^{n_2} \in W$

$$\dots$$

$b^{n_1} a b^{n_2} a \dots \in \overrightarrow{W} = L$, contradiction.

Deterministic Büchi Automata are not closed under complement

Theorem 5 *There exists a DBA A such that no DBA recognizes the language $\Sigma^\omega \setminus \mathcal{L}(A)$.*

$\Sigma = \{a, b\}$ and $L = \{\alpha \in \Sigma^\omega \mid \#_a(\alpha) < \infty\} = \Sigma^* b^\omega$.

Let $V = \Sigma^* a$. There exists a DFA A such that $\mathcal{L}(A) = V$.

There exists a deterministic Büchi automaton B such that $\mathcal{L}(A) = \overrightarrow{V}$

But $\Sigma^\omega \setminus \overrightarrow{V} = L$ which cannot be recognized by any DBA.

Büchi Automata and S1S

Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet.

Any finite word $w \in \Sigma^*$ induces the *infinite* sets $p_a = \{p \mid w(p) = a\}$.

- $x \leq y$: x is less than y,
- $s(x) = y$: y is the successor of x,
- $p_a(x)$: a occurs at position x in w

Remember that $\leq$ and s can be defined one from another.

Problem Statement

Let $\mathcal{L}(\varphi) = \{w \mid \mathfrak{m}_w \models \varphi\}$

A language $L \subseteq \Sigma^\omega$ is said to be S1S-*definable* iff there exists a S1S formula φ such that $L = \mathcal{L}(\varphi)$.

1. Given a Büchi automaton A build an S1S formula φ_A such that $\mathcal{L}(A) = \mathcal{L}(\varphi)$
2. Given an S1S formula φ build a Büchi automaton A_φ such that $\mathcal{L}(A) = \mathcal{L}(\varphi)$

The Büchi recognizable and S1S-definable languages coincide

From Automata to Formulae

Let $A = \langle S, I, T, F \rangle$ with $S = \{s_1, \ldots, s_p\}$, and $\Sigma = \{0,1\}^m$.

Build $\Phi_A(X_1, \ldots, X_m)$ such that $\forall w \in \Sigma^* \;.\; w \in \mathcal{L}(A) \iff \mathfrak{m}_w \models \Phi_A$

$\Phi_A(X_1, \ldots, X_m) = \exists Y_1 \ldots \exists Y_p \;.\; \Phi_S(\vec{Y}) \wedge \Phi_I(\vec{Y}) \wedge \Phi_T(\vec{Y}, \vec{X}) \wedge \Phi_F(\vec{Y})$

$$\Phi_F(\vec{Y}) = \forall x \exists y \;.\; x \leq y \wedge x \neq y \wedge \bigvee_{s_i \in F} Y_i(y)$$

From Formulae to Automata

Let $\Phi(X_1, \ldots, X_p, x_{p+1}, \ldots, x_m)$ be a S1S formula.

Build an automaton A_Φ such that $\forall w \in \Sigma^* \;.\; w \in \mathcal{L}(A) \iff [\![\Phi]\!]^{\mathfrak{m}_w}_{\iota_w} = \text{true}$

Let $\Phi(X_1, X_2, x_3, x_4)$ be:

1. $X_1(x_3)$
2. $x_3 \leq x_4$
3. $X_1 = X_2$

From Formulae to Automata

A_Φ is built by induction on the structure of Φ:

- for $\Phi = \phi_1 \wedge \phi_2$ we have $\mathcal{L}(A_\Phi) = \mathcal{L}(A_{\phi_1}) \cap \mathcal{L}(A_{\phi_2})$
- for $\Phi = \phi_1 \vee \phi_2$ we have $\mathcal{L}(A_\Phi) = \mathcal{L}(A_{\phi_1}) \cup \mathcal{L}(A_{\phi_2})$
- for $\Phi = \neg\phi$ we have $\mathcal{L}(A_\Phi) = \overline{\mathcal{L}(A_\phi)}$ (requires complementation)
- for $\Phi = \exists X_i \, . \, \phi$, we have $\mathcal{L}(A_\Phi) = pr_i(\mathcal{L}(A_\phi))$.

Consequences

Theorem 6 *A language $L \subseteq \Sigma^\omega$ is definable in S1S iff it is ω-recognizable.*

Corollary 2 *The SAT problem for S1S is decidable.*

Muller and Rabin Word Automata

Muller Automata

Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet.

Definition 1 *A* Muller automaton *over* Σ *is* $A = \langle S, s_0, T, \mathcal{F} \rangle$*, where:*

- S *is the finite* set of states
- $s_0 \in S$ *is the* initial state
- $T : S \times \Sigma \mapsto S$ *is the* transition table
- $\mathcal{F} \subseteq 2^S$ *is the set of* accepting sets

Notice that Muller automata are deterministic and complete by definition.

Acceptance Condition

A *run* of a Muller automaton is defined over an infinite word $w : \alpha_1 \alpha_2 \ldots$ as an infinite sequence of states $\pi \ : \ s_0 s_1 s_2 \ldots$ such that:

- $T(s_i, \alpha_{i+1}) = s_{i+1}$, for all $i \in \mathbb{N}$.

Let $\inf(\pi) = \{s \mid s$ appears infinitely often on $\pi\}$.

Run π of A is said to be *accepting* iff $\inf(\pi) \in \mathcal{F}$.

$L \subseteq \Sigma^\omega$ is *Muller-recognizable* iff there exists a MA A such that $L = \mathcal{L}(A)$.

Exercises

Exercise 1 *Let* $\Sigma = \{a, b\}$ *and* $A = \langle S, s_a, T, \mathcal{F} \rangle$, *where:*

- $S = \{s_a, s_b\}$,
- $T(s_a, a) = s_a$, $T(s_a, b) = s_b$, $T(s_b, a) = s_a$ *and* $T(s_b, b) = s_b$,
- $\mathcal{F} = \{\{s_a, s_b\}\}$

What is $\mathcal{L}(A)$*? What if* A *was Büchi with* $F = \{s_a, s_b\}$*?*

Exercise 2 *Build a Muller automaton recognizing the following language:*
$\Sigma = \{a, b\}$, $L = (a + b)^* a^\omega$

Closure Properties

Theorem 7 *The class of Muller-recognizable languages is closed under union, intersection and complement.*

Let $A = \langle S, s_0, T, \mathcal{F} \rangle$ be a Muller automaton.

Define $B = \langle S, s_0, T, 2^S \setminus \mathcal{F} \rangle$.

We have $\mathcal{L}(B) = \Sigma^\omega \setminus \mathcal{L}(A)$.

Closure Properties

Let $A_i = \langle S_i, s_{0,i}, T_i, \mathcal{F}_i \rangle$, $i = 1, 2$ be Muller automata.

Define $B = \langle S, s_0, T, \mathcal{F} \rangle$ where:

- $S = S_1 \times S_2$,
- $s_0 = \langle s_{0,1}, s_{0,2} \rangle$,
- $T(\langle s_1, s_2 \rangle, a) = \langle T(s_1, a), T(s_2, a) \rangle$
- $\mathcal{F} = \{\{\langle s_1, s_1' \rangle, \ldots, \langle s_k, s_k' \rangle\} \mid \{s_1, \ldots, s_k\} \in \mathcal{F}_1$ or $\{s_1', \ldots, s_k'\} \in \mathcal{F}_2\}$

We have $\mathcal{L}(B) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$.

For intersection it is enough to set

$$\mathcal{F} = \{\{\langle s_1, s_1' \rangle, \ldots, \langle s_k, s_k' \rangle\} \mid \{s_1, \ldots, s_k\} \in \mathcal{F}_1 \text{ and } \{s_1', \ldots, s_k'\} \in \mathcal{F}_2\}$$

Deterministic Büchi $\subseteq$ Muller

Theorem 8 *For each deterministic Büchi automaton A there exists a Muller automaton B such that $\mathcal{L}(A) = \mathcal{L}(B)$*

Let $A = \langle S, \{s_0\}, T, F\rangle$ be a deterministic Büchi automaton.

Define $B = \langle S, s_0, T, \{G \in 2^S \mid G \cap F \neq \emptyset\}\rangle$

Muller $\subseteq$ Non-deterministic Büchi

Theorem 9 *For each Muller automaton A there exists a non-deterministic Büchi automaton B such that $\mathcal{L}(A) = \mathcal{L}(B)$.*

Let $A = (S, s_0, T, \mathcal{F})$ be a Muller automaton, with $\mathcal{F} = \{F_1, \ldots, F_n\}$. Then B simulates A and guesses the accepting set F_i.

We introduce finite memory to accumulate F_i states. The Büchi automaton guesses when all the states outside F_i are finished.

When the memory is full we reset it to $\emptyset$, to ensure that we see F_i states again and again.

Muller $\subseteq$ Non-deterministic Büchi

Define the Büchi automaton $B = (S_B, s_0, T_B, F_B)$ where:

- $S_B = S \cup (S \times 2^S \times \{1, \ldots, n\})$
- $F_B = \{(s, \emptyset, i) \mid s \in S,\ i \in \{1, \ldots, n\}\}$
- T_B is defined as follows:
  - $(s, \alpha, t) \in T_B$ and $(s, \alpha, (t, \emptyset, i)) \in T_B$ if $T(s, \alpha) = t$
  - $((s, Q, i), \alpha, (t, Q \cup \{t\}, i)) \in T_B$ if $T(s, \alpha) = t$ and $Q \cup \{t\} \subset F_i$
  - $((s, Q, i), \alpha, (t, \emptyset, i)) \in T_B$ if $T(s, \alpha) = t$ and $Q \cup \{t\} = F_i$

Now we prove that $\mathcal{L}(A) = \mathcal{L}(B)$.

Characterization of Muller-recognizable languages

A language $L \subseteq \Sigma^\omega$ is Muller-recognizable iff L is a Boolean combination of sets $\overrightarrow{W}$, $W \subseteq \Sigma^*$ recognizable, i.e. $L = \bigcup_i \left(\bigcap_j \overrightarrow{W_{ij}} \cap \bigcap_k (\Sigma^\omega \setminus \overrightarrow{W_{ik}}) \right)$.

"$\Leftarrow$" Any set $\overrightarrow{W_{ij}}$ is recognized by a deterministic Büchi automaton, hence also by a Muller automaton.

"$\Rightarrow$" Let $A = \langle S, s_0, T, \mathcal{F} \rangle$ be a Muller automaton recognizing L.

Let $A_q = \langle S, s_0, T, \{q\} \rangle$, $q \in S$, and $W_q = \mathcal{L}(A_q)$.

$$L = \bigcup_{Q \in \mathcal{F}} \left(\bigcap_{q \in Q} \overrightarrow{W_q} \cap \bigcap_{q \in S \setminus Q} (\Sigma^\omega \setminus \overrightarrow{W_q}) \right)$$

Rabin Word Automata

Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet.

Definition 2 *A* Rabin automaton *over* Σ *is* $A = \langle S, s_0, T, \Omega \rangle$*, where:*

- S *is the finite* set of states
- $s_0 \in S$ *is the* initial state
- $T : S \times \Sigma \mapsto S$ *is the* transition table
- $\Omega = \{\langle N_1, P_1 \rangle, \ldots, \langle N_k, P_k \rangle\}$ *is the set of* accepting pairs*,* $N_i, P_i \subseteq S$*.*

Run π of A is said to be *accepting* iff

$$\inf(\pi) \cap N_i = \emptyset \text{ and } \inf(\pi) \cap P_i \neq \emptyset$$

for some $1 \leq i \leq k$.

Exercises

Exercise 3 *Let* $\Sigma = \{a, b\}$. *Write down a Rabin automaton for the following languages:*

1. $L = \{w \mid a \text{ occurs infinitely often and } b \text{ occurs finitely often in } w\}$

2. $L = \{w \mid a \text{ occurs finitely often and } b \text{ occurs infinitely often in } w\}$

From Rabin to Muller

Given a Rabin automaton $A = \langle S, s_0, T, \Omega \rangle$, there exists a Muller automaton B such that $\mathcal{L}(A) = \mathcal{L}(B)$

Let $\Omega = \{\langle N_1, P_1 \rangle, \ldots, \langle N_k, P_k \rangle\}$.

Let $A_i = \langle S, s_0, T, P_i \rangle$ and $B_i = \langle S, s_0, T, N_i \rangle$ be DFA.

$$\mathcal{L}(A) = \bigcup_{i=1}^{k} \left(\overrightarrow{\mathcal{L}(A_i)} \cap (\Sigma^\omega \setminus \overrightarrow{\mathcal{L}(B_i)}) \right)$$

From Rabin to Muller (a constructive approach)

Given a Rabin automaton $A = \langle S, s_0, T, \Omega \rangle$, such that

$$\Omega = \{\langle N_1, P_1 \rangle, \ldots, \langle N_k, P_k \rangle\}$$

let $B = \langle S, s_0, T, \mathcal{F} \rangle$ be the Muller automaton, where

$$\mathcal{F} = \{F \subseteq S \mid F \cap N_i = \emptyset \text{ and } F \cap P_i \neq \emptyset \text{ for some } 1 \leq i \leq k\}$$

Exercise 4 *Let $A = \langle S, s_0, T, \{Q_1, \ldots, Q_t\} \rangle$ be a Muller automaton. Consider the Rabin automaton $A' = \langle S, s_0, T, \Omega \rangle$ where*

$$\Omega = \{(S \setminus Q_1, Q_1), \ldots, (S \setminus Q_t, Q_t)\}$$

Give an example of A such that $\mathcal{L}(A) \neq \mathcal{L}(A')$.

From Muller to Rabin

Given a Muller automaton $A = \langle S, s_0, T, \mathcal{F} \rangle$, there exists a Rabin automaton B such that $\mathcal{L}(A) = \mathcal{L}(B)$

Let $\mathcal{F} = \{Q_1, \ldots, Q_k\}$

Let $B = \langle S', s'_0, T', \Omega' \rangle$ where:

- $S' = 2^{Q_1} \times \ldots \times 2^{Q_k} \times S$
- $s'_0 = \langle \emptyset, \ldots, \emptyset, s_0 \rangle$

From Muller to Rabin

- $T'(\langle S_1, \ldots, S_k, s\rangle, a) = \langle S'_1, \ldots, S'_k, s'\rangle$ where:
  - $s' = T(s, a)$
  - $S'_i = \emptyset$ if $S_i = Q_i$, $1 \leq i \leq k$
  - $S'_i = (S_i \cup \{s'\}) \cap Q_i$, $1 \leq i \leq k$

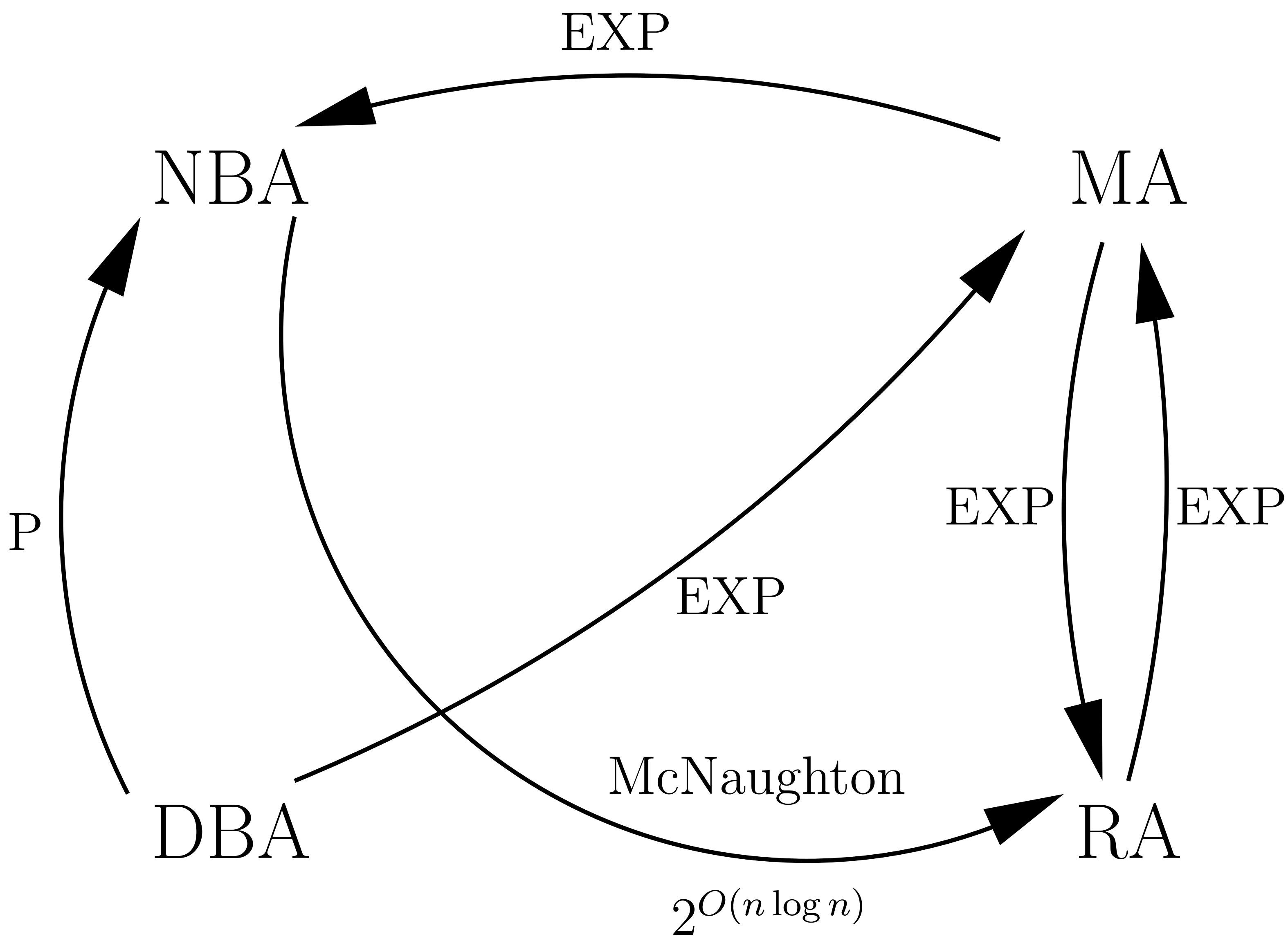
- $P_i = \{\langle S_1, \ldots, S_i, \ldots, S_k, s\rangle \mid S_i = Q_i\}$, $1 \leq i \leq k$

- $N_i = \{\langle S_1, \ldots, S_i, \ldots, S_k, s\rangle \mid s \notin Q_i\}$, $1 \leq i \leq k$

Exercises

Exercise 5 *Build a Rabin automaton for the language:* $\Sigma = \{a, b\}$,
$L = \{w \mid \textit{if } a \textit{ occurs infinitely often then } b \textit{ occurs infinitely often in } w\}$

The Big Picture



ω-Regular Languages

If $X \subseteq \Sigma^*$ and $Y \subseteq \Sigma^\omega$

$$\begin{aligned} XY &= \{xy \mid x \in X,\ y \in Y\} \in \Sigma^\omega \\ X^\omega &= \{x_0 x_1 \ldots \mid x_0, x_1, \ldots \in X \setminus \{\epsilon\}\} \\ X^\infty &= X^* \cup X^\omega \end{aligned}$$

The class of *ω-regular languages* $\mathcal{R}^\infty(\Sigma)$ is the smallest class of languages $L \subseteq \Sigma^\infty$ such that:

- $\emptyset \in \mathcal{R}^\infty(\Sigma)$ and $\{a\} \in \mathcal{R}^\infty(\Sigma)$, for all $a \in \Sigma$
- if $X, Y \in \mathcal{R}^\infty(\Sigma)$ then $X \cup Y \in \mathcal{R}^\infty(\Sigma)$
- for each $X \subseteq \Sigma^*$ and $Y \subseteq \Sigma^\infty$, if $X, Y \in \mathcal{R}^\infty(\Sigma)$ then $XY \in \mathcal{R}^\infty(\Sigma)$
- for each $X \subseteq \Sigma^*$, if $X \in \mathcal{R}^\infty(\Sigma)$ then $X^*, X^\omega \in \mathcal{R}^\infty(\Sigma)$

Star Free ω-Languages

The class of *star-free ω-languages* is the smallest class $SF^{\infty}(\Sigma)$ of languages $L \in \Sigma^*$ such that:

- $\emptyset, \{a\} \in SF^{\infty}(\Sigma),\ a \in \Sigma$
- if $X, Y \in SF^{\infty}(\Sigma)$ then $X \cup Y, \overline{X} \in SF^{\infty}(\Sigma)$
- if $X \subseteq \Sigma^*,\ X \in SF(\Sigma),\ Y \in SF^{\infty}(\Sigma)$ then $XY \in SF^{\infty}(\Sigma)$

Example 1

- *if $B \subset \Sigma$, then $\Sigma^* B \Sigma^\omega$ is star-free*
- *if $\Sigma = \{a, b\}$, then $(ab)^\omega = \overline{b\Sigma^\omega \cup \Sigma^* aa\Sigma^\omega \cup \Sigma^* bb\Sigma^\omega}$ is star-free*

The Big Picture