

Automata on Infinite Words

Definition of Büchi Automata

Let $\Sigma = \{a, b, \dots\}$ be a finite alphabet.

A *non-deterministic Büchi automaton* (NBA) over Σ is a tuple $A = \langle S, I, T, F \rangle$, where:

- S is a finite set of *states*,
- $I \subseteq S$ is a set of *initial states*,
- $T \subseteq S \times \Sigma \times S$ is a *transition relation*,
- $F \subseteq S$ is a set of *final states*.

Acceptance Condition

A *run* of a Büchi automaton is defined over an infinite word $w : \alpha_1\alpha_2\ldots$ as an infinite sequence of states $\pi : s_0s_1s_2\ldots$ such that:

- $s_0 \in I$ and
- $(s_i, \alpha_{i+1}, s_{i+1}) \in T$, for all $i \in \mathbb{N}$.

$$\boxed{\text{inf}(\pi) = \{s \mid s \text{ appears infinitely often on } \pi\}}$$

Run π of A is said to be *accepting* iff $\text{inf}(\pi) \cap F \neq \emptyset$.

The language of A , denoted $\mathcal{L}(A)$, is the set of all words accepted by A .

A language $L \subseteq \Sigma^\omega$ is *ω -recognizable* if there exists a Büchi automaton A such that $L = \mathcal{L}(A)$.

Examples

Let $\Sigma = \{0, 1\}$. Define Büchi automata for the following languages:

1. $L = \{\alpha \in \Sigma^\omega \mid 0 \text{ occurs in } \alpha \text{ exactly once}\}$
2. $L = \{\alpha \in \Sigma^\omega \mid \text{after each } 0 \text{ in } \alpha \text{ there is } 1\}$
3. $L = \{\alpha \in \Sigma^\omega \mid \alpha \text{ contains finitely many } 1\text{'s}\}$
4. $L = (01)^*\Sigma^\omega$
5. $L = \{\alpha \in \Sigma^\omega \mid 0 \text{ occurs on all even positions in } \alpha\}$

Büchi Characterization Theorem

Lemma 1 *If $L \subseteq \Sigma^*$ is a recognizable language, there exists a DFA $A = \langle S, \{s_0\}, T, F \rangle$ such that s_0 has no incoming transitions and $L = \mathcal{L}(A)$.*

Given $W \subseteq \Sigma^*$, define $W^\omega = \{w_0w_1 \dots \mid w_i \in W, i \geq 0\}$

Theorem 1 *Let $W, V \subseteq \Sigma^*$ be recognizable languages. Then the languages W^ω and $V \cdot W^\omega$ are ω -recognizable.*

Büchi Characterization Theorem

Let $A = \langle S, I, T, F \rangle$ be a Büchi automaton and $s, s' \in S$ be two states.

Let $W_{s,s'} = \{w \in \Sigma^* \mid s \xrightarrow{w} s'\}$.

The language $W_{s,s'} \subseteq \Sigma^*$ is recognizable, for any $s, s' \in S$.

Theorem 2 *An ω -language $L \subseteq \Sigma^\omega$ is ω -recognizable iff L is a finite union of ω -languages $V \cdot W^\omega$, where $V, W \subseteq \Sigma^*$ are recognizable languages.*

Proof idea: $L = \bigcup_{s \in I, s' \in F} W_{s,s'} \cdot W_{s',s'}^\omega$

Corollary 1 *Any non-empty Büchi-recognizable language contains an ultimately periodic word of the form $uvvv \dots$*

The Emptiness Problem

Theorem 3 *Given a Büchi automaton A , $\mathcal{L}(A) \neq \emptyset$ iff there exist $u, v \in \Sigma^*$, $|u|, |v| \leq \|A\|$, such that $uv^\omega \in \mathcal{L}(A)$.*

In practical terms, A is non-empty iff there exists a state s which is reachable both from an initial state and from itself.

Q: Is the membership problem decidable for Büchi automata?

Closure Properties

Closure under **union** and **projection** are like in the finite automata case.

Intersection is a bit special.

Complementation of non-deterministic Büchi automata is a complex result.

Deterministic Büchi automata are not closed under complement.

Closure under Intersection

Let $A_1 = \langle S_1, I_1, T_1, F_1 \rangle$ and $A_2 = \langle S_2, I_2, T_2, F_2 \rangle$

Build $A_\cap = \langle S, I, T, F \rangle$:

- $S = S_1 \times S_2 \times \{1, 2, 3\}$,
- $I = I_1 \times I_2 \times \{1\}$,
- the definition of T is the following:
 - $((s_1, s'_1, 1), a, (s_2, s'_2, 1)) \in T$ iff $(s_i, a, s'_i) \in T_i$, $i = 1, 2$ and $s_1 \notin F_1$
 - $((s_1, s'_1, 1), a, (s_2, s'_2, 2)) \in T$ iff $(s_i, a, s'_i) \in T_i$, $i = 1, 2$ and $s_1 \in F_1$
 - $((s_1, s'_1, 2), a, (s_2, s'_2, 2)) \in T$ iff $(s_i, a, s'_i) \in T_i$, $i = 1, 2$ and $s'_1 \notin F_2$
 - $((s_1, s'_1, 2), a, (s_2, s'_2, 3)) \in T$ iff $(s_i, a, s'_i) \in T_i$, $i = 1, 2$ and $s'_1 \in F_2$
 - $((s_1, s'_1, 3), a, (s_2, s'_2, 1)) \in T$ iff $(s_i, a, s'_i) \in T_i$, $i = 1, 2$
- $F = S_1 \times S_2 \times \{3\}$

Deterministic Büchi Automata

ω -languages recognized by NBA \supset ω -languages recognized by DBA

Q: Why classical subset construction does not work for Büchi automata?

Let $A = \langle S, I, T, F \rangle$ and $A_d = \langle 2^S, \{I\}, T_d, \{Q \mid Q \cap F \neq \emptyset\} \rangle$.

Let $u_0u_1u_2 \dots \in \mathcal{L}(A)$ be an infinite word. In A_d this gives:

$$I \xrightarrow{u_0} Q_1 \xrightarrow{u_1} Q_2 \xrightarrow{u_2} \dots$$

where each $Q_i \cap F$. However this does not necessarily correspond to an accepting path in A !

Deterministic Büchi Automata

Let $W \subseteq \Sigma^*$. Define $\vec{W} = \{\alpha \in \Sigma^\omega \mid \alpha(0, n) \in W \text{ for infinitely many } n\}$

Theorem 4 *A language $L \subseteq \Sigma^\omega$ is recognizable by a deterministic Büchi automaton iff there exists a recognizable language $W \subseteq \Sigma^*$ such that $L = \vec{W}$.*

If $L = \mathcal{L}(A)$ then $W = \mathcal{L}(A')$ where A' is the DFA with the same definition as A , and with the **finite acceptance condition**.

Deterministic Büchi Automata

Theorem 5 *There exists an ω -recognizable language that can be recognized by no deterministic Büchi automaton.*

$$\Sigma = \{a, b\} \text{ and } L = \{\alpha \in \Sigma^\omega \mid \#_a(\alpha) < \infty\} = \Sigma^*b^\omega.$$

Suppose $L = \overrightarrow{W}$ for some $W \subseteq \Sigma^*$.

$$b^\omega \in L \Rightarrow b^{n_1} \in W$$

$$b^{n_1}ab^\omega \in L \Rightarrow b^{n_1}ab^{n_2} \in W$$

...

$$b^{n_1}ab^{n_2}a \dots \in \overrightarrow{W} = L, \text{ contradiction.}$$

Deterministic Büchi Automata are not closed under complement

Theorem 6 *There exists a DBA A such that no DBA recognizes the language $\Sigma^\omega \setminus \mathcal{L}(A)$.*

$$\Sigma = \{a, b\} \text{ and } L = \{\alpha \in \Sigma^\omega \mid \#_a(\alpha) < \infty\} = \Sigma^*b^\omega.$$

Let $V = \Sigma^*a$. There exists a DFA A such that $\mathcal{L}(A) = V$.

There exists a deterministic Büchi automaton B such that $\mathcal{L}(B) = \overrightarrow{V}$

But $\Sigma^\omega \setminus \overrightarrow{V} = L$ which cannot be recognized by any DBA.

Büchi Automata and S1S

Let $\Sigma = \{a, b, \dots\}$ be a finite alphabet.

Any finite word $w \in \Sigma^*$ induces the *infinite* sets $p_a = \{p \mid w(p) = a\}$.

- $x \leq y$: x is less than y ,
- $s(x) = y$: y is the successor of x ,
- $p_a(x)$: a occurs at position x in w

Remember that \leq and s can be defined one from another.

Problem Statement

Let $\mathcal{L}(\varphi) = \{w \mid \mathfrak{m}_w \models \varphi\}$

A language $L \subseteq \Sigma^*$ is said to be S1S-*definable* iff there exists a S1S formula φ such that $L = \mathcal{L}(\varphi)$.

1. Given a Büchi automaton A build an S1S formula φ_A such that $\mathcal{L}(A) = \mathcal{L}(\varphi)$
2. Given an S1S formula φ build a Büchi automaton A_φ such that $\mathcal{L}(A) = \mathcal{L}(\varphi)$

The Büchi recognizable and S1S-definable languages coincide

From Automata to Formulae

Let $A = \langle S, I, T, F \rangle$ with $S = \{s_1, \dots, s_p\}$, and $\Sigma = \{0, 1\}^m$.

Build $\Phi_A(X_1, \dots, X_m)$ such that $\forall w \in \Sigma^* . w \in \mathcal{L}(A) \iff \mathfrak{m}_w \models \Phi_A$

$$\Phi_A(X_1, \dots, X_m) = \exists Y_1 \dots \exists Y_p . \Phi_S(\vec{Y}) \wedge \Phi_I(\vec{Y}) \wedge \Phi_T(\vec{Y}, \vec{X}) \wedge \Phi_F(\vec{Y})$$

$$\Phi_F(\vec{Y}) = \forall x \exists y . x \leq y \wedge x \neq y \wedge \bigvee_{s_i \in F} Y_i(y)$$

From Formulae to Automata

Let $\Phi(X_1, \dots, X_p, x_{p+1}, \dots, x_m)$ be a S1S formula.

Build an automaton A_Φ such that $\forall w \in \Sigma^* . w \in \mathcal{L}(A) \iff \llbracket \Phi \rrbracket_{\iota_w}^{\mathfrak{m}_w} = \text{true}$

Let $\Phi(X_1, X_2, x_3, x_4)$ be:

1. $X_1(x_3)$
2. $x_3 \leq x_4$
3. $X_1 = X_2$

From Formulae to Automata

A_Φ is built by induction on the structure of Φ :

- for $\Phi = \phi_1 \wedge \phi_2$ we have $\mathcal{L}(A_\Phi) = \mathcal{L}(A_{\phi_1}) \cap \mathcal{L}(A_{\phi_2})$
- for $\Phi = \phi_1 \vee \phi_2$ we have $\mathcal{L}(A_\Phi) = \mathcal{L}(A_{\phi_1}) \cup \mathcal{L}(A_{\phi_2})$
- for $\Phi = \neg\phi$ we have $\mathcal{L}(A_\Phi) = \overline{\mathcal{L}(A_\phi)}$ (requires complementation)
- for $\Phi = \exists X_i . \phi$, we have $\mathcal{L}(A_\Phi) = pr_i(\mathcal{L}(A_\phi))$.

Consequences

Theorem 7 *A language $L \subseteq \Sigma^\omega$ is definable in S1S iff it is ω -recognizable.*

Corollary 2 *The SAT problem for S1S is decidable.*

Muller and Rabin Word Automata

Muller Automata

Let $\Sigma = \{a, b, \dots\}$ be a finite alphabet.

Definition 1 A Muller automaton over Σ is $A = \langle S, s_0, T, \mathcal{F} \rangle$, where:

- S is the finite set of states
- $s_0 \in S$ is the initial state
- $T : S \times \Sigma \mapsto S$ is the transition table
- $\mathcal{F} \subseteq 2^S$ is the set of accepting sets

Notice that Muller automata are deterministic and complete by definition.

Acceptance Condition

A *run* of a Muller automaton is defined over an infinite word $w : \alpha_1\alpha_2\ldots$ as an infinite sequence of states $\pi : s_0s_1s_2\ldots$ such that:

- $T(s_i, \alpha_{i+1}) = s_{i+1}$, for all $i \in \mathbb{N}$.

Let $\text{inf}(\pi) = \{s \mid s \text{ appears infinitely often on } \pi\}$.

Run π of A is said to be *accepting* iff $\text{inf}(\pi) \in \mathcal{F}$.

$L \subseteq \Sigma^\omega$ is *Muller-recognizable* iff there exists a MA A such that $L = \mathcal{L}(A)$.

Exercises

Exercise 1 Let $\Sigma = \{a, b\}$ and $A = \langle S, s_a, T, \mathcal{F} \rangle$, where:

- $S = \{s_a, s_b\}$,
- $T(s_a, a) = s_a$, $T(s_a, b) = s_b$, $T(s_b, a) = s_a$ and $T(s_b, b) = s_b$,
- $\mathcal{F} = \{\{s_a, s_b\}\}$

What is $\mathcal{L}(A)$? What if A was Büchi with $F = \{s_a, s_b\}$?

Exercise 2 Build a Muller automaton recognizing the following language:

$$\Sigma = \{a, b\}, L = (a + b)^* a^\omega$$

Closure Properties

Theorem 8 *The class of Muller-recognizable languages is closed under union, intersection and complement.*

Let $A = \langle S, s_0, T, \mathcal{F} \rangle$ be a Muller automaton.

Define $B = \langle S, s_0, T, 2^S \setminus \mathcal{F} \rangle$.

We have $\mathcal{L}(B) = \Sigma^\omega \setminus \mathcal{L}(A)$.

Closure Properties

Let $A_i = \langle S_i, s_{0,i}, T_i, \mathcal{F}_i \rangle$, $i = 1, 2$ be Muller automata.

Define $B = \langle S, s_0, T, \mathcal{F} \rangle$ where:

- $S = S_1 \times S_2$,
- $s_0 = \langle s_{0,1}, s_{0,2} \rangle$,
- $T(\langle s_1, s_2 \rangle, a) = \langle T(s_1, a), T(s_2, a) \rangle$
- $\mathcal{F} = \{ \{ \langle s_1, s'_1 \rangle, \dots, \langle s_k, s'_k \rangle \} \mid \{s_1, \dots, s_k\} \in \mathcal{F}_1 \text{ or } \{s'_1, \dots, s'_k\} \in \mathcal{F}_2 \}$

We have $\mathcal{L}(B) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$.

For intersection it is enough to set

$$\mathcal{F} = \{ \{ \langle s_1, s'_1 \rangle, \dots, \langle s_k, s'_k \rangle \} \mid \{s_1, \dots, s_k\} \in \mathcal{F}_1 \text{ and } \{s'_1, \dots, s'_k\} \in \mathcal{F}_2 \}$$

Deterministic Büchi \subseteq Muller

Theorem 9 *For each deterministic Büchi automaton A there exists a Muller automaton B such that $\mathcal{L}(A) = \mathcal{L}(B)$*

Let $A = \langle S, \{s_0\}, T, F \rangle$ be a deterministic Büchi automaton.

Define $B = \langle S, s_0, T, \{G \in 2^S \mid G \cap F \neq \emptyset\} \rangle$

Muller \subseteq Non-deterministic Büchi

Theorem 10 *For each Muller automaton A there exists a non-deterministic Büchi automaton B such that $\mathcal{L}(A) = \mathcal{L}(B)$.*

Let $A = (S, s_0, T, \mathcal{F})$ be a Muller automaton, with $\mathcal{F} = \{F_1, \dots, F_n\}$. Then B simulates A and **guesses** the accepting set F_i .

We introduce finite memory to accumulate F_i states. The Büchi automaton **guesses** when all the states outside F_i are finished.

When the memory is full we reset it to \emptyset , to ensure that we see F_i states again and again.

Muller \subseteq Non-deterministic Büchi

Define the Büchi automaton $B = (S_B, s_0, T_B, F_B)$ where:

- $S_B = S \cup (S \times 2^S \times \{1, \dots, n\})$
- $F_B = \{(s, \emptyset, i) \mid s \in S, i \in \{1, \dots, n\}\}$
- T_B is defined as follows:
 - $(s, \alpha, t) \in T_B$ and $(s, \alpha, (t, \emptyset, i)) \in T_B$ if $T(s, \alpha) = t$
 - $((s, Q, i), \alpha, (t, Q \cup \{t\}, i)) \in T_B$ if $T(s, \alpha) = t$ and $Q \cup \{t\} \subset F_i$
 - $((s, Q, i), \alpha, (t, \emptyset, i)) \in T_B$ if $T(s, \alpha) = t$ and $Q \cup \{t\} = F_i$

Now we prove that $\mathcal{L}(A) = \mathcal{L}(B)$.

Characterization of Muller-recognizable languages

A language $L \subseteq \Sigma^\omega$ is Muller-recognizable iff L is a Boolean combination of sets \overrightarrow{W} , $W \subseteq \Sigma^*$, i.e. $L = \bigcup_i \left(\bigcap_j \overrightarrow{W}_{ij} \cap \bigcap_k (\Sigma^\omega \setminus \overrightarrow{W}_{ik}) \right)$.

“ \Leftarrow ” Any set \overrightarrow{W}_{ij} is recognized by a deterministic Büchi automaton, hence also by a Muller automaton.

“ \Rightarrow ” Let $A = \langle S, s_0, T, \mathcal{F} \rangle$ be a Muller automaton recognizing L .

Let $A_q = \langle S, s_0, T, \{q\} \rangle$, $q \in S$, and $W_q = \mathcal{L}(A_q)$.

$$L = \bigcup_{Q \in \mathcal{F}} \left(\bigcap_{q \in Q} \overrightarrow{W}_q \cap \bigcap_{q \in S \setminus Q} (\Sigma^\omega \setminus \overrightarrow{W}_q) \right)$$

Rabin Word Automata

Let $\Sigma = \{a, b, \dots\}$ be a finite alphabet.

Definition 2 A **Rabin automaton** over Σ is $A = \langle S, s_0, T, \Omega \rangle$, where:

- S is the finite set of states
- $s_0 \in S$ is the initial state
- $T : S \times \Sigma \mapsto S$ is the transition table
- $\Omega = \{\langle N_1, P_1 \rangle, \dots, \langle N_k, P_k \rangle\}$ is the set of accepting pairs, $N_i, P_i \subseteq S$.

Run π of A is said to be *accepting* iff

$$\inf(\pi) \cap N_i = \emptyset \text{ and } \inf(\pi) \cap P_i \neq \emptyset$$

for some $1 \leq i \leq k$.

Exercises

***Exercise 3** Let $\Sigma = \{a, b\}$. Write down a Rabin automaton for the following languages:*

- 1. $L = \{w \mid a \text{ occurs infinitely often and } b \text{ occurs finitely often in } w\}$*
- 2. $L = \{w \mid a \text{ occurs finitely often and } b \text{ occurs infinitely often in } w\}$*

From Rabin to Muller

Given a Rabin automaton $A = \langle S, s_0, T, \Omega \rangle$, there exists a Muller automaton B such that $\mathcal{L}(A) = \mathcal{L}(B)$

Let $\Omega = \{\langle N_1, P_1 \rangle, \dots, \langle N_k, P_k \rangle\}$.

Let $A_i = \langle S, s_0, T, P_i \rangle$ and $B_i = \langle S, s_0, T, N_i \rangle$ be DFA.

$$\mathcal{L}(A) = \bigcup_{i=1}^k \left(\overrightarrow{\mathcal{L}(A_i)} \cap (\Sigma^\omega \setminus \overrightarrow{\mathcal{L}(B_i)}) \right)$$

From Rabin to Muller (a constructive approach)

Given a Rabin automaton $A = \langle S, s_0, T, \Omega \rangle$, such that

$$\Omega = \{ \langle N_1, P_1 \rangle, \dots, \langle N_k, P_k \rangle \}$$

let $B = \langle S, s_0, T, \mathcal{F} \rangle$ be the Muller automaton, where

$$\mathcal{F} = \{ F \subseteq S \mid F \cap N_i = \emptyset \text{ and } F \cap P_i \neq \emptyset \text{ for some } 1 \leq i \leq k \}$$

Exercise 4 Let $A = \langle S, s_0, T, \{Q_1, \dots, Q_t\} \rangle$ be a Muller automaton.

Consider the Rabin automaton $A' = \langle S, s_0, T, \Omega \rangle$ where

$$\Omega = \{ (S \setminus Q_1, Q_1), \dots, (S \setminus Q_t, Q_t) \}$$

Give an example of A such that $\mathcal{L}(A) \neq \mathcal{L}(A')$.

From Muller to Rabin

Given a Muller automaton $A = \langle S, s_0, T, \mathcal{F} \rangle$, there exists a Rabin automaton B such that $\mathcal{L}(A) = \mathcal{L}(B)$

Let $\mathcal{F} = \{Q_1, \dots, Q_k\}$

Let $B = \langle S', s'_0, T', \Omega' \rangle$ where:

- $S' = 2^{Q_1} \times \dots \times 2^{Q_k} \times S$
- $s'_0 = \langle \emptyset, \dots, \emptyset, s_0 \rangle$

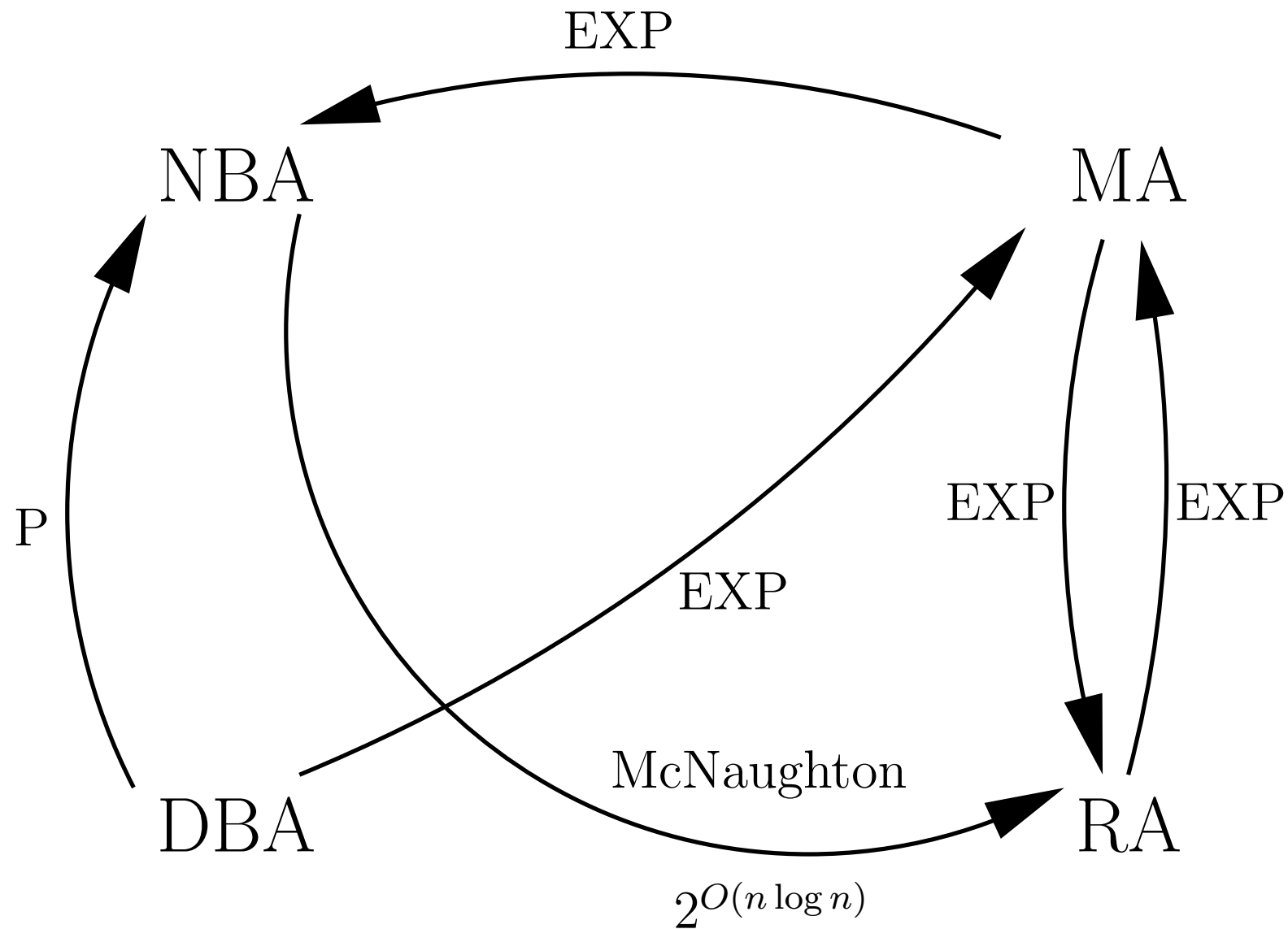
From Muller to Rabin

- $T'(\langle S_1, \dots, S_k, s \rangle, a) = \langle S'_1, \dots, S'_k, s' \rangle$ where:
 - $s' = T(s, a)$
 - $S'_i = \emptyset$ if $S_i = Q_i$, $1 \leq i \leq k$
 - $S'_i = (S_i \cup \{s'\}) \cap Q_i$, $1 \leq i \leq k$
- $P_i = \{ \langle S_1, \dots, S_i, \dots, S_k, s \rangle \mid S_i = Q_i \}, 1 \leq i \leq k$
- $N_i = \{ \langle S_1, \dots, S_i, \dots, S_k, s \rangle \mid s \notin Q_i \}, 1 \leq i \leq k$

Exercises

***Exercise 5** Build a Rabin automaton for the language: $\Sigma = \{a, b\}$,
 $L = \{w \mid \text{if } a \text{ occurs infinitely often then } b \text{ occurs infinitely often in } w\}$*

The Big Picture



ω -Regular Languages

If $X \subseteq \Sigma^*$ and $Y \subseteq \Sigma^\omega$

$$XY = \{xy \mid x \in X, y \in Y\} \in \Sigma^\omega$$

$$X^\omega = \{x_0x_1\ldots \mid x_0, x_1, \ldots \in X \setminus \{\epsilon\}\}$$

$$X^\infty = X^* \cup X^\omega$$

The class of *ω -regular languages* $\mathcal{R}^\infty(\Sigma)$ is the smallest class of languages $L \subseteq \Sigma^\infty$ such that:

- $\emptyset \in \mathcal{R}^\infty(\Sigma)$ and $\{a\} \in \mathcal{R}^\infty(\Sigma)$, for all $a \in \Sigma$
- if $X, Y \in \mathcal{R}^\infty(\Sigma)$ then $X \cup Y \in \mathcal{R}^\infty(\Sigma)$
- for each $X \subseteq \Sigma^*$ and $Y \subseteq \Sigma^\infty$, if $X, Y \in \mathcal{R}^\infty(\Sigma)$ then $XY \in \mathcal{R}^\infty(\Sigma)$
- for each $X \subseteq \Sigma^*$, if $X \in \mathcal{R}^\infty(\Sigma)$ then $X^*, X^\omega \in \mathcal{R}^\infty(\Sigma)$

Star Free ω -Languages

The class of *star-free ω -languages* is the smallest class $SF^\infty(\Sigma)$ of languages $L \in \Sigma^*$ such that:

- $\emptyset, \{a\} \in SF^\infty(\Sigma), a \in \Sigma$
- if $X, Y \in SF^\infty(\Sigma)$ then $X \cup Y, \overline{X} \in SF^\infty(\Sigma)$
- if $X \subseteq \Sigma^*, X \in SF(\Sigma), Y \in SF^\infty(\Sigma)$ then $XY \in SF^\infty(\Sigma)$

Example 1

- if $B \subset \Sigma$, then $\Sigma^* B \Sigma^\omega$ is star-free
- if $\Sigma = \{a, b\}$, then $(ab)^\omega = \overline{b\Sigma^\omega \cup \Sigma^* aa\Sigma^\omega \cup \Sigma^* bb\Sigma^\omega}$ is star-free

The Big Picture

