

# Automata on Infinite Words

## Definition of Büchi Automata

Let  $\Sigma = \{a, b, \dots\}$  be a finite alphabet.

A *non-deterministic Büchi automaton* (NBA) over  $\Sigma$  is a tuple  $A = \langle S, I, T, F \rangle$ , where:

- $S$  is a finite set of *states*,
- $I \subseteq S$  is a set of *initial states*,
- $T \subseteq S \times \Sigma \times S$  is a *transition relation*,
- $F \subseteq S$  is a set of *final states*.

## Acceptance Condition

A *run* of a Büchi automaton is defined over an infinite word  $w : \alpha_1\alpha_2\ldots$  as an infinite sequence of states  $\pi : s_0s_1s_2\ldots$  such that:

- $s_0 \in I$  and
- $(s_i, \alpha_{i+1}, s_{i+1}) \in T$ , for all  $i \in \mathbb{N}$ .

$$\inf(\pi) = \{s \mid s \text{ appears infinitely often on } \pi\}$$

Run  $\pi$  of  $A$  is said to be *accepting* iff  $\inf(\pi) \cap F \neq \emptyset$ .

The language of  $A$ , denoted  $\mathcal{L}(A)$ , is the set of all words accepted by  $A$ .

A language  $L \subseteq \Sigma^\omega$  is  *$\omega$ -recognizable* if there exists a Büchi automaton  $A$  such that  $L = \mathcal{L}(A)$ .

## Examples

Let  $\Sigma = \{0, 1\}$ . Define Büchi automata for the following languages:

1.  $L = \{\alpha \in \Sigma^\omega \mid 0 \text{ occurs in } \alpha \text{ exactly once}\}$
2.  $L = \{\alpha \in \Sigma^\omega \mid \text{after each } 0 \text{ in } \alpha \text{ there is } 1\}$
3.  $L = \{\alpha \in \Sigma^\omega \mid \alpha \text{ contains finitely many } 1\text{'s}\}$
4.  $L = (01)^*\Sigma^\omega$
5.  $L = \{\alpha \in \Sigma^\omega \mid 0 \text{ occurs on all even positions in } \alpha\}$

## The Büchi Characterization Theorems

**Lemma 1** *If  $L \subseteq \Sigma^*$  is a recognizable language, there exists a DFA  $A = \langle S, \{s_0\}, T, F \rangle$  such that  $s_0$  has no incoming transitions and  $L = \mathcal{L}(A)$ .*

**Theorem 1** *Let  $W, V \subseteq \Sigma^*$  be recognizable languages. Then the language  $V \cdot W^\omega$  is  $\omega$ -recognizable.*

Is it possible that, for some non-recognizable language  $W \subseteq \Sigma^*$ ,  $W^\omega$  is  $\omega$ -recognizable?

## First Büchi Characterization Theorem

Let  $A = \langle S, I, T, F \rangle$  be a Büchi automaton and  $s, s' \in S$  be two states.

Let  $W_{s,s'} = \{w \in \Sigma^* \mid s \xrightarrow{w} s'\}$ .

The language  $W_{s,s'} \subseteq \Sigma^*$  is recognizable, for any  $s, s' \in S$ .

**Theorem 2** *An  $\omega$ -language  $L \subseteq \Sigma^\omega$  is  $\omega$ -recognizable iff  $L$  is a finite union of  $\omega$ -languages  $V \cdot W^\omega$ , where  $V, W \subseteq \Sigma^*$  are recognizable languages.*

Proof idea:  $L = \bigcup_{s \in I, s' \in F} W_{s,s'} \cdot W_{s',s'}^\omega$

## Second Büchi Characterization Theorem

**Theorem 3** *An  $\omega$ -language  $L \subseteq \Sigma^\omega$  is  $\omega$ -recognizable iff  $L$  is a finite union of  $\omega$ -languages  $V \cdot W^\omega$ , where  $V, W \subseteq \Sigma^*$  are recognizable languages such that  $W \cdot W \subseteq W$ .*

Proof idea:  $\mathcal{L}(A) = \bigcup_{s \in I, s' \in F} W_{s,s'} W_{s',s'}^\omega$  and  $W_{s',s'} \cdot W_{s',s'} \subseteq W_{s',s'}$  for all  $s' \in F$

**Corollary 1** *Any non-empty Büchi-recognizable language contains an ultimately periodic word of the form  $uvvv \dots$*

# Closure Properties

Closure under **union** and **projection** are like in the finite automata case.

**Intersection** is a bit special.

**Complementation** of non-deterministic Büchi automata is a complex result.

**Deterministic Büchi automata are not closed under complement.**



## Closure under Intersection

Let  $A_1 = \langle S_1, I_1, T_1, F_1 \rangle$  and  $A_2 = \langle S_2, I_2, T_2, F_2 \rangle$

Build  $A_\cap = \langle S, I, T, F \rangle$ :

- $S = S_1 \times S_2 \times \{1, 2, 3\}$ ,
- $I = I_1 \times I_2 \times \{1\}$ ,
- the definition of  $T$  is the following:
  - $((s_1, s'_1, 1), a, (s_2, s'_2, 1)) \in T$  iff  $(s_i, a, s'_i) \in T_i$ ,  $i = 1, 2$  and  $s_1 \notin F_1$
  - $((s_1, s'_1, 1), a, (s_2, s'_2, 2)) \in T$  iff  $(s_i, a, s'_i) \in T_i$ ,  $i = 1, 2$  and  $s_1 \in F_1$
  - $((s_1, s'_1, 2), a, (s_2, s'_2, 2)) \in T$  iff  $(s_i, a, s'_i) \in T_i$ ,  $i = 1, 2$  and  $s'_1 \notin F_2$
  - $((s_1, s'_1, 2), a, (s_2, s'_2, 3)) \in T$  iff  $(s_i, a, s'_i) \in T_i$ ,  $i = 1, 2$  and  $s'_1 \in F_2$
  - $((s_1, s'_1, 3), a, (s_2, s'_2, 1)) \in T$  iff  $(s_i, a, s'_i) \in T_i$ ,  $i = 1, 2$
- $F = S_1 \times S_2 \times \{3\}$

## The Emptiness Problem

**Theorem 4** *Given a Büchi automaton  $A$ ,  $\mathcal{L}(A) \neq \emptyset$  iff there exist  $u, v \in \Sigma^*$ ,  $|u|, |v| \leq \|A\|$ , such that  $uv^\omega \in \mathcal{L}(A)$ .*

In practical terms,  $A$  is non-empty iff there exists a state  $s$  which is reachable both from an initial state and from itself.

**Q:** Is the membership problem decidable for Büchi automata?

## Deterministic Büchi Automata

$\omega$ -languages recognized by NBA  $\supset$   $\omega$ -languages recognized by DBA

**Q:** Why classical subset construction does not work for Büchi automata?

Let  $A = \langle S, I, T, F \rangle$  and  $A_d = \langle 2^S, \{I\}, T_d, \{Q \mid Q \cap F \neq \emptyset\} \rangle$ .

Let  $u_0u_1u_2 \dots \in \mathcal{L}(A)$  be an infinite word. In  $A_d$  this gives:

$$I \xrightarrow{u_0} Q_1 \xrightarrow{u_1} Q_2 \xrightarrow{u_2} \dots$$

where each  $Q_i \cap F$ . However this does not necessarily correspond to an accepting path in  $A$ !

## Deterministic Büchi Automata

Let  $W \subseteq \Sigma^*$ . Define  $\vec{W} = \{\alpha \in \Sigma^\omega \mid \alpha(0, n) \in W \text{ for infinitely many } n\}$

**Theorem 5** *A language  $L \subseteq \Sigma^\omega$  is recognizable by a deterministic Büchi automaton iff there exists a recognizable language  $W \subseteq \Sigma^*$  such that  $L = \vec{W}$ .*

If  $L = \mathcal{L}(A)$  then  $W = \mathcal{L}(A')$  where  $A'$  is the DFA with the same definition as  $A$ , and with the **finite acceptance condition**.

## Deterministic Büchi Automata

**Theorem 6** *There exists an  $\omega$ -recognizable language that can be recognized by no deterministic Büchi automaton.*

$$\Sigma = \{a, b\} \text{ and } L = \{\alpha \in \Sigma^\omega \mid \#_a(\alpha) < \infty\} = \Sigma^*b^\omega.$$

Suppose  $L = \overrightarrow{W}$  for some  $W \subseteq \Sigma^*$ .

$$b^\omega \in L \Rightarrow b^{n_1} \in W$$

$$b^{n_1}ab^\omega \in L \Rightarrow b^{n_1}ab^{n_2} \in W$$

...

$$b^{n_1}ab^{n_2}a \dots \in \overrightarrow{W} = L, \text{ contradiction.}$$

## Deterministic Büchi Automata are not closed under complement

**Theorem 7** *There exists a DBA  $A$  such that no DBA recognizes the language  $\Sigma^\omega \setminus \mathcal{L}(A)$ .*

$$\Sigma = \{a, b\} \text{ and } L = \{\alpha \in \Sigma^\omega \mid \#_a(\alpha) < \infty\} = \Sigma^*b^\omega.$$

Let  $V = \Sigma^*a$ . There exists a DFA  $A$  such that  $\mathcal{L}(A) = V$ .

There exists a deterministic Büchi automaton  $B$  such that  $\mathcal{L}(B) = \overrightarrow{V}$

But  $\Sigma^\omega \setminus \overrightarrow{V} = L$  which cannot be recognized by any DBA.

## Büchi Automata and S1S

Let  $\Sigma = \{a, b, \dots\}$  be a finite alphabet.

Any finite word  $w \in \Sigma^*$  induces the *infinite* sets  $p_a = \{p \mid w(p) = a\}$ .

- $x \leq y$  :  $x$  is less than  $y$ ,
- $s(x) = y$  :  $y$  is the successor of  $x$ ,
- $p_a(x)$  :  $a$  occurs at position  $x$  in  $w$

Remember that  $\leq$  and  $s$  can be defined one from another.

## Problem Statement

Let  $\mathcal{L}(\varphi) = \{w \mid \mathfrak{m}_w \models \varphi\}$

A language  $L \subseteq \Sigma^*$  is said to be S1S-*definable* iff there exists a S1S formula  $\varphi$  such that  $L = \mathcal{L}(\varphi)$ .

1. Given a Büchi automaton  $A$  build an S1S formula  $\varphi_A$  such that  $\mathcal{L}(A) = \mathcal{L}(\varphi)$
2. Given an S1S formula  $\varphi$  build a Büchi automaton  $A_\varphi$  such that  $\mathcal{L}(A) = \mathcal{L}(\varphi)$

The Büchi recognizable and S1S-definable languages coincide



## From Automata to Formulae

Let  $A = \langle S, I, T, F \rangle$  with  $S = \{s_1, \dots, s_p\}$ , and  $\Sigma = \{0, 1\}^m$ .

Build  $\Phi_A(X_1, \dots, X_m)$  such that  $\forall w \in \Sigma^* . w \in \mathcal{L}(A) \iff \mathfrak{m}_w \models \Phi_A$

$$\Phi_A(X_1, \dots, X_m) = \exists Y_1 \dots \exists Y_p . \Phi_S(\vec{Y}) \wedge \Phi_I(\vec{Y}) \wedge \Phi_T(\vec{Y}, \vec{X}) \wedge \Phi_F(\vec{Y})$$

$$\Phi_F(\vec{Y}) = \forall x \exists y . x \leq y \wedge x \neq y \wedge \bigvee_{s_i \in F} Y_i(y)$$

## Consequences

**Theorem 8** *A language  $L \subseteq \Sigma^\omega$  is definable in S1S iff it is Büchi recognizable.*

**Corollary 2** *The SAT problem for S1S is decidable.*

# Muller and Rabin Word Automata

# Muller Automata

Let  $\Sigma = \{a, b, \dots\}$  be a finite alphabet.

**Definition 1** A Muller automaton over  $\Sigma$  is  $A = \langle S, s_0, T, \mathcal{F} \rangle$ , where:

- $S$  is the finite set of states
- $s_0 \in S$  is the initial state
- $T : S \times \Sigma \mapsto S$  is the transition table
- $\mathcal{F} \subseteq 2^S$  is the set of accepting sets

Notice that Muller automata are deterministic and complete by definition.

## Acceptance Condition

A *run* of a Muller automaton is defined over an infinite word  $w : \alpha_1\alpha_2\ldots$  as an infinite sequence of states  $\pi : s_0s_1s_2\ldots$  such that:

- $T(s_i, \alpha_{i+1}) = s_{i+1}$ , for all  $i \in \mathbb{N}$ .

Let  $\text{inf}(\pi) = \{s \mid s \text{ appears infinitely often on } \pi\}$ .

Run  $\pi$  of  $A$  is said to be *accepting* iff  $\text{inf}(\pi) \in \mathcal{F}$ .

$L \subseteq \Sigma^\omega$  is *Muller-recognizable* iff there exists a MA  $A$  such that  $L = \mathcal{L}(A)$ .

## Exercise

Let  $\Sigma = \{a, b\}$  and  $A = \langle S, s_0, T, \mathcal{F} \rangle$ , where:

- $S = \{s_0, s_1\}$ ,
- $T(s_0, a) = s_0$ ,  $T(s_0, b) = s_1$ ,  $T(s_1, a) = s_0$  and  $T(s_1, b) = s_1$ ,
- $\mathcal{F} = \{\{s_0, s_1\}\}$

What is  $\mathcal{L}(A)$ ? What if  $A$  was Büchi with  $F = \{s_0, s_1\}$ ?

## Deterministic Büchi $\subseteq$ Muller

**Theorem 9** *For each deterministic Büchi automaton  $A$  there exists a Muller automaton  $B$  such that  $\mathcal{L}(A) = \mathcal{L}(B)$*

Let  $A = \langle S, \{s_0\}, T, F \rangle$  be a deterministic Büchi automaton.

Define  $B = \langle S, s_0, T, \{G \in 2^S \mid G \cap F \neq \emptyset\} \rangle$

## Muller $\subseteq$ Non-deterministic Büchi

**Theorem 10** *For each Muller automaton  $A$  there exists a non-deterministic Büchi automaton  $B$  such that  $\mathcal{L}(A) = \mathcal{L}(B)$ .*

Let  $A = (S, s_0, T, \mathcal{F})$  be a Muller automaton, with  $\mathcal{F} = \{F_1, \dots, F_n\}$ . Then  $B$  simulates  $A$  and **guesses** the accepting set  $F_i$ .

We introduce finite memory to accumulate  $F_i$  states. The Büchi automaton **guesses** when all the states outside  $F_i$  are finished.

When the memory is full we reset it to  $\emptyset$ , to ensure that we see  $F_i$  states again and again.



## Muller $\subseteq$ Non-deterministic Büchi

Define the Büchi automaton  $B = (S_B, s_0, T_B, F_B)$  where:

- $S_B = S \cup (S \times 2^S \times \{1, \dots, n\})$
- $F_B = \{(s, \emptyset, i) \mid s \in S, i \in \{1, \dots, n\}\}$
- $T_B$  is defined as follows:
  - $(s, \alpha, t) \in T_B$  and  $(s, \alpha, (t, \emptyset, i)) \in T_B$  if  $T(s, \alpha) = t$
  - $((s, Q, i), \alpha, (t, Q \cup \{t\}, i)) \in T_B$  if  $T(s, \alpha) = t$  and  $Q \cup \{t\} \subset F_i$
  - $((s, Q, i), \alpha, (t, \emptyset, i)) \in T_B$  if  $T(s, \alpha) = t$  and  $Q \cup \{t\} = F_i$

Now we prove that  $\mathcal{L}(A) = \mathcal{L}(B)$ .

## Closure Properties

**Theorem 11** *The class of Muller-recognizable languages is closed under union, intersection and complement.*

Let  $A = \langle S, s_0, T, \mathcal{F} \rangle$  be a Muller automaton.

Define  $B = \langle S, s_0, T, 2^S \setminus \mathcal{F} \rangle$ .

We have  $\mathcal{L}(B) = \Sigma^\omega \setminus \mathcal{L}(A)$ .

## Closure Properties

Let  $A_i = \langle S_i, s_{0,i}, T_i, \mathcal{F}_i \rangle$ ,  $i = 1, 2$  be Muller automata.

Define  $B = \langle S, s_0, T, \mathcal{F} \rangle$  where:

- $S = S_1 \times S_2$ ,
- $s_0 = \langle s_{0,1}, s_{0,2} \rangle$ ,
- $T(\langle s_1, s_2 \rangle, a) = \langle T(s_1, a), T(s_2, a) \rangle$
- $\mathcal{F} = \{ \{ \langle s_1, s'_1 \rangle, \dots, \langle s_k, s'_k \rangle \} \mid \{s_1, \dots, s_k\} \in \mathcal{F}_1 \text{ or } \{s'_1, \dots, s'_k\} \in \mathcal{F}_2 \}$

We have  $\mathcal{L}(B) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$ .

For intersection it is enough to set

$$\mathcal{F} = \{ \{ \langle s_1, s'_1 \rangle, \dots, \langle s_k, s'_k \rangle \} \mid \{s_1, \dots, s_k\} \in \mathcal{F}_1 \text{ and } \{s'_1, \dots, s'_k\} \in \mathcal{F}_2 \}$$

## Characterization of Muller-recognizable languages

A language  $L \subseteq \Sigma^\omega$  is Muller-recognizable iff  $L$  is a Boolean combination of sets  $\overrightarrow{W}$ ,  $W \subseteq \Sigma^*$ , i.e.  $L = \bigcup_i \left( \bigcap_j \overrightarrow{W}_{ij} \cap \bigcap_k (\Sigma^\omega \setminus \overrightarrow{W}_{ik}) \right)$ .

“ $\Leftarrow$ ” Any set  $\overrightarrow{W}_{ij}$  is recognized by a deterministic Büchi automaton, hence also by a Muller automaton.

“ $\Rightarrow$ ” Let  $A = \langle S, s_0, T, \mathcal{F} \rangle$  be a Muller automaton recognizing  $L$ .

Let  $A_q = \langle S, s_0, T, \{q\} \rangle$ ,  $q \in S$ , and  $W_q = \mathcal{L}(A_q)$ .

$$L = \bigcup_{Q \in \mathcal{F}} \left( \bigcap_{q \in Q} \overrightarrow{W}_q \cap \bigcap_{q \in S \setminus Q} (\Sigma^\omega \setminus \overrightarrow{W}_q) \right)$$

# Rabin Word Automata

Let  $\Sigma = \{a, b, \dots\}$  be a finite alphabet.

**Definition 2** A **Rabin automaton** over  $\Sigma$  is  $A = \langle S, s_0, T, \Omega \rangle$ , where:

- $S$  is the finite set of states
- $s_0 \in S$  is the initial state
- $T : S \times \Sigma \mapsto S$  is the transition table
- $\Omega = \{\langle N_1, P_1 \rangle, \dots, \langle N_k, P_k \rangle\}$  is the set of accepting pairs,  $N_i, P_i \subseteq S$ .

Run  $\pi$  of  $A$  is said to be *accepting* iff

$$\inf(\pi) \cap N_i = \emptyset \text{ and } \inf(\pi) \cap P_i \neq \emptyset$$

for some  $1 \leq i \leq k$ .

## From Rabin to Muller

Given a Rabin automaton  $A = \langle S, s_0, T, \Omega \rangle$ , there exists a Muller automaton  $B$  such that  $\mathcal{L}(A) = \mathcal{L}(B)$

Let  $\Omega = \{ \langle N_1, P_1 \rangle, \dots, \langle N_k, P_k \rangle \}$ .

Let  $A_i = \langle S, s_0, T, P_i \rangle$ , and  $B_i = \langle S, s_0, T, N_i \rangle$ .

$$\mathcal{L}(A) = \bigcup_{i=1}^k \left( \overrightarrow{\mathcal{L}(A_i)} \cap (\Sigma^\omega \setminus \overrightarrow{\mathcal{L}(B_i)}) \right)$$

## From Rabin to Muller (2)

Given a Rabin automaton  $A = \langle S, s_0, T, \Omega \rangle$ , such that

$$\Omega = \{ \langle N_1, P_1 \rangle, \dots, \langle N_k, P_k \rangle \}$$

let  $B = \langle S, s_0, T, \mathcal{F} \rangle$  be the Muller automaton, where

$$\mathcal{F} = \{ F \subseteq S \mid F \cap N_i = \emptyset \text{ and } F \cap P_i \neq \emptyset \text{ for some } 1 \leq i \leq k \}$$

## From Muller to Rabin

Given a Muller automaton  $A = \langle S, s_0, T, \mathcal{F} \rangle$ , there exists a Rabin automaton  $B$  such that  $\mathcal{L}(A) = \mathcal{L}(B)$

Let  $\mathcal{F} = \{Q_1, \dots, Q_k\}$

Let  $B = \langle S', s'_0, T', \Omega' \rangle$  where:

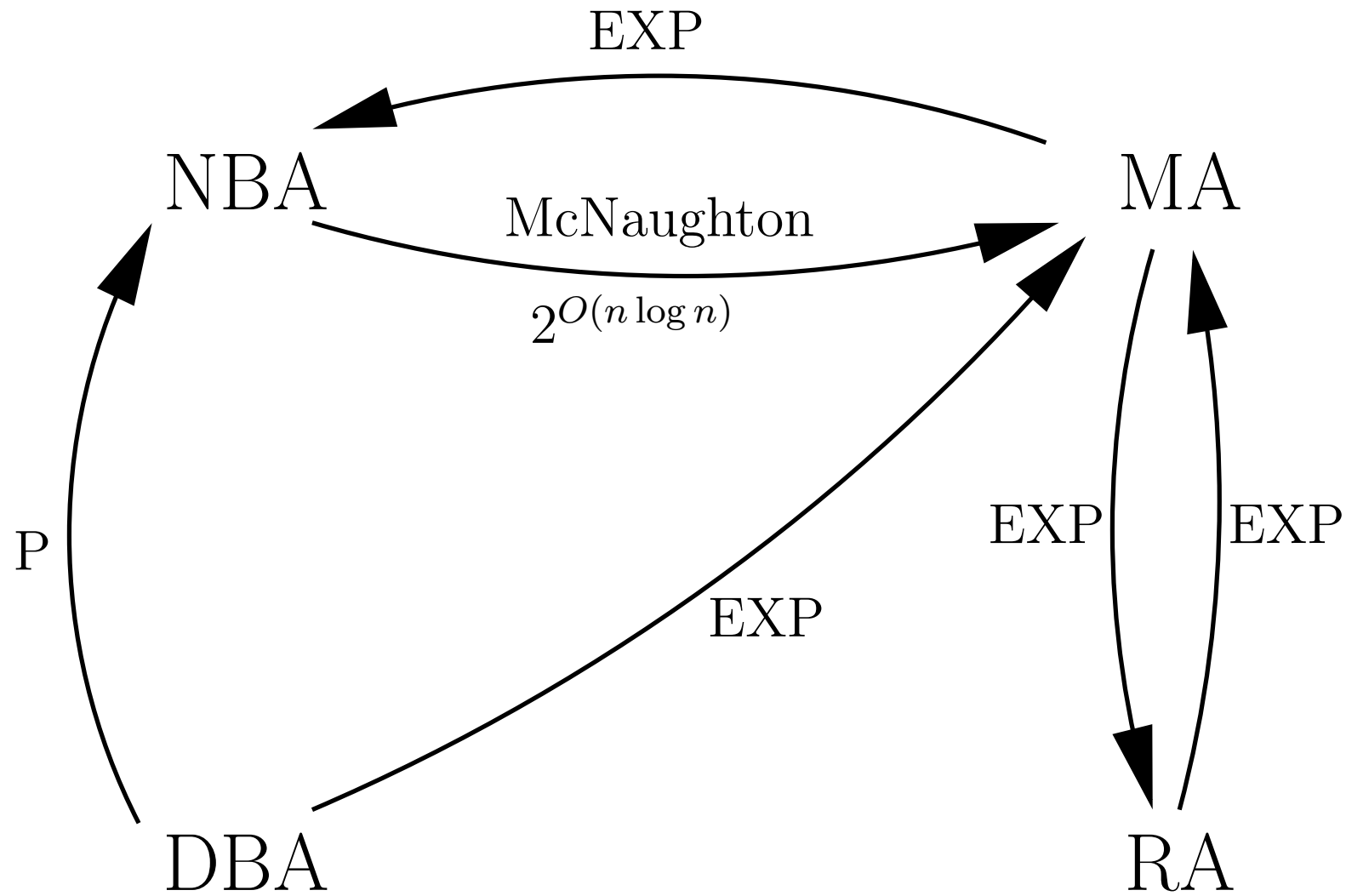
- $S' = 2^{Q_1} \times \dots \times 2^{Q_k} \times S$
- $s'_0 = \langle \emptyset, \dots, \emptyset, s_0 \rangle$



## From Muller to Rabin

- $T'(\langle S_1, \dots, S_k, s \rangle, a) = \langle S'_1, \dots, S'_k, s' \rangle$  where:
  - $s' = T(s, a)$
  - $S'_i = \emptyset$  if  $S_i = Q_i$ ,  $1 \leq i \leq k$
  - $S'_i = (S_i \cup \{s'\}) \cap Q_i$ ,  $1 \leq i \leq k$
- $P_i = \{ \langle S_1, \dots, S_i, \dots, S_k, s \rangle \mid S_i = Q_i \}, 1 \leq i \leq k$
- $N_i = \{ \langle S_1, \dots, S_i, \dots, S_k, s \rangle \mid s \notin Q_i \}, 1 \leq i \leq k$

## The Big Picture



## Exercises

Write down Muller automata for the following languages:

1.  $\Sigma = \{a, b\}, L = (a + b)^* a^\omega$
2.  $\Sigma = \{a, b, c\}, L =$   
 $\{w \mid \text{if } a \text{ occurs infinitely often then } b \text{ occurs infinitely often in } w\}$

Let  $A = \langle S, s_0, T, \{Q_1, \dots, Q_t\} \rangle$  be a Muller automaton. Consider the Rabin automaton  $A' = \langle S, s_0, T, \Omega \rangle$  where

$$\Omega = \{(S \setminus Q_1, Q_1), \dots, (S \setminus Q_t, Q_t)\}$$

Give an example of  $A$  such that  $\mathcal{L}(A) \neq \mathcal{L}(A')$ .