Infinite Games

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Build Correct HW/SW Systems

- Use logic to specify correctness properties, e.g.:
  - *every job sent to the printer is eventually printed*
  - *two jobs do not overlap (only one job is printed at a time)*
  - *a job that is canceled will be interupted*

  These are conditions on infinite sequences (system runs), and can be specified by automata and logical formulas.

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  These are conditions on infinite sequences (system runs), and can be specified by automata and logical formulas.
- Given a logical specification, we can do either:
  - VERIFICATION: prove that a given system satisfies the specification
  - SYNTHESIS: build a system that satisfies the specification

Example: Elevator

- Aim: build controller that moves elevator of 10 floor building
- Environment: Passengers pressing buttons to (1) call elevator and (2) request floor
- System state:
  1. Set of requested floor numbers: $\{0,1\}^{10}$
  2. Current position of lift: $\{1,\ldots,10\}$
  3. Indicator whose turn is next (assuming lift and passengers act in alternation) $\{0,1\}$

Infinite Games

Two players:

1. Controller is Player 0
2. Passengers are Player 1

A play of a game is an infinite sequence of states of elevator transition system, where the two players choose moves alternatively.

How does the transition system look like?

- State space: $\{0,1\}^{10} \times \{1,\ldots,10\} \times \{0,1\}$
- Transitions:
  - Player 0: $(r_1 \ldots r_{10}, j, 0) \to [r'_1 \ldots r'_{10}, j', 1]$ s.t. $r_j = 1$, $r'_j = 0$ and $\forall_{i \neq j} r_i = r'_i$. Actions: open/closes doors and move lift
  - Player 1: $[r_1 \ldots r_{10}, j, 1] \to (r'_1 \ldots r'_{10}, j', 0)$ s.t. $j = j'$, $\forall i : r_i \leq r'_i$ Actions: request floors

Desired Properties

- Every requested floor is eventually reached
- Floors along the way are served if requested
- If no floor is request, elevator goes to ground floor
- ...

These are conditions on infinite sequences!

Player 0 (controller) wins the play if all conditions are satisfied independent of the choices Player 1 makes. This corresponds to finding a winning strategy for Player 0 in an infinite game.

Our Aim

Solution of the Synthesis Problem

1. Decide whether there exists such a winning strategy - Realizability Problem
2. If “yes”, then construct the system - Synthesis Problem

Main result:

The synthesis problem is algorithmically solvable for finite-state systems with respect to specifications given as ω-automata or linear-time temporal logic.

Other: Nicer and more intuitive proofs for logics over trees

Outline

Terminology

Terminology

Two-player games between Player 0 and 1

An infinite game $\langle G, \phi \rangle$ consists of

- a game graph G and
- a winning condition ϕ.

G defines the "playground", in which the two players compete.

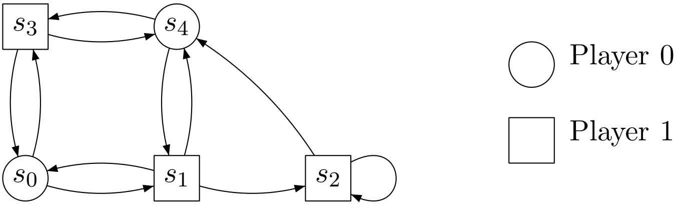
ϕ defines which plays are won by Player 0.

If a play does not satisfy ϕ, then Player 1 wins on this play.

Game Graphs

A game graph is a tuple $G = \langle S, S_0, T \rangle$ where:

- S is a finite set of states,
- $S_0 \subseteq S$ is the set of Player-0 states ($S_1 = S \setminus S_0$ are the Player-1 states),
- $T \subseteq S \times S$ is a transition relation. We assume that each state has at least one successor.



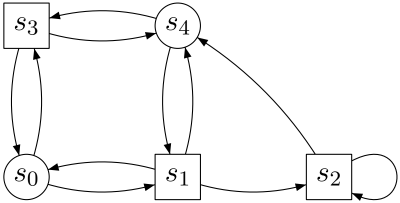
Plays

A play is an infinite sequence of states $\rho = s_0 s_1 s_2 \cdots \in S^\omega$ such that for all $i \geq 0$ $\langle s_i, s_{i+1} \rangle \in T$.

It starts in s_0 and it is built up as follows:

If $s_i \in S_0$, then Player 0 chooses an edge starting in s_i, otherwise Player 1 picks such an edge.

Intuitively, a token is moved from state to state via edges: From S_0-states Player 0 moves the token, from S_1-states Player 1 moves the token.



Winning Condition

The winning condition describes the plays won by Player 0.

A winning condition or winning objective ϕ is a subset of plays, i.e., $\phi \subseteq S^\omega$.

We use logical conditions (e.g., LTL formulas) or automata theoretic acceptance conditions to describe ϕ.

Example:

- $\Box\Diamond s$ for some state $s \in S$
- All plays that stay within a safe region $F \subseteq S$ are in ϕ.
- Given a priority function $p : S \to \{0, 1, \ldots, d\}$, all plays in which the smallest priority visited is even.

Games are named after their winning condition, e.g., Safety game, Reachability game, LTL game, Parity game,...

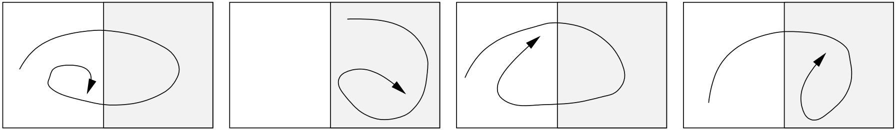
Types of Games

Given a play ρ, we define

- $\mathrm{Occ}(\rho) = \{s \in S \mid \exists i \geq 0 : s_i = s\}$
- $\mathrm{Inf}(\rho) = \{s \in S \mid \forall i \geq 0 \exists j > i : s_j = s\}$

Given a set $F \subseteq S$,

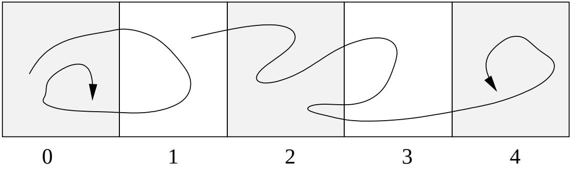
Reachability Game	$\phi = \{\rho \in S^\omega \mid \mathrm{Occ}(\rho) \cap F \neq \emptyset\}$
Safety Game	$\phi = \{\rho \in S^\omega \mid \mathrm{Occ}(\rho) \subseteq F\}$
Büchi Game	$\phi = \{\rho \in S^\omega \mid \mathrm{Inf}(\rho) \cap F \neq \emptyset\}$
Co-Büchi Game	$\phi = \{\rho \in S^\omega \mid \mathrm{Inf}(\rho) \subseteq F\}$



Types of Games

Given a priority function $p : S \to \{0, 1, \ldots, d\}$ or an LTL formula φ

Weak-Parity Game	$\phi = \{\rho \in S^\omega \mid \min_{s \in \mathrm{Occ}(\rho)} p(s) \text{ is even}\}$
Parity Game	$\phi = \{\rho \in S^\omega \mid \min_{s \in \mathrm{Inf}(\rho)} p(s) \text{ is even}\}$
LTL Game	$\phi = \{\rho \in S^\omega \mid \rho \models \varphi\}$



We will refer to the type of a game and give F, p, or φ instead of defining ϕ.

We will also talk about Muller and Rabin games.

Strategies

A strategy for Player 0 from state s is a (partial) function

$$f : S^* S_0 \to S$$

specifying for any sequence of states $s_0, s_1, \dots s_k$ with $s_0 = s$ and $s_k \in S_0$ a successor state s_j such that $(s_k, s_j) \in T$.
A play $\rho = s_0 s_1 \dots$ is compatible with strategy f if for all $s_i \in S_0$ we have that $s_{i+1} = f(s_0 s_1 \dots s_i)$.
(Definitions for Player 1 are analogous.)

Given strategies f and g from s for Player 0 and 1, respectively, we denote by $G_{f,g}$ the (unique) play that is compatible with f and g.

Winning Strategies and Regions

Given a game (G, ϕ) with $G = (S, S_0, E)$, a strategy f for Player 0 from s is called a winning strategy if for all Player-1 strategies g from s, $G_{f,g} \in \phi$ holds. Analogously, a Player-1 strategy g is winning if for all Player-0 strategies f, $G_{f,g} \notin \phi$ holds.

Player 0 (resp. 1) wins from s if s/he has a winning strategy from s.

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Player 0 (resp. 1) wins from s if s/he has a winning strategy from s.

The winning regions of Player 0 and 1 are the sets

$$W_0 = \{s \in S \mid \text{Player 0 wins from } s\}$$

$$W_1 = \{s \in S \mid \text{Player 1 wins from } s\}$$

Note each state s belongs at most to W_0 or W_1. Otherwise pick winning strategies f and g from s for Player 0 and 1, respectively, then $G_{f,g} \in \phi$ and $G_{f,g} \notin \phi$: Contradiction.

Questions About Games

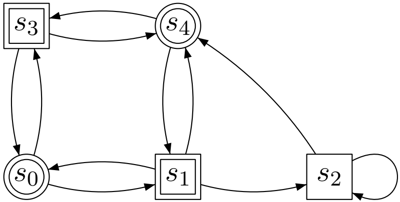
Solve a game (G, ϕ) with $G = (S, S_0, T)$:

1. Decide for each state $s \in S$ if $s \in W_0$.
2. If yes, construct a suitable winning strategy from s.

Further interesting question:

- Optimize construction of winning strategy (e.g., time complexity)
- Optimize parameters of winning strategy (e.g., size of memory)

Example



Safety game (G, F) with $F = \{s_0, s_1, s_3, s_4\}$, i.e., $\text{Occ}(\rho) \subseteq F$

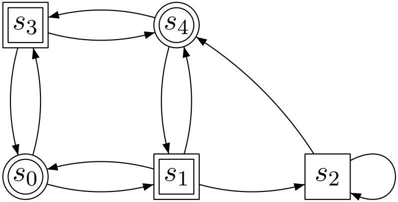
A winning strategy for Player 0 (from state s_0 and s_4):

- From s_0 choose s_3 and from s_4 choose s_3

A winning strategy for Player 1 (from state s_1 and s_2):

- From s_1 choose s_2, from s_2 choose s_4

Example



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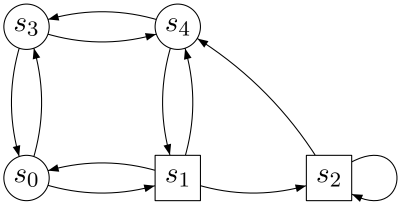
- From s_1 choose s_2, from s_2 choose s_4

$W_0 = \{s_0, s_3, s_4\}$, $W_1 = \{s_1, s_2\}$

Another Example



LTL game (G, φ) with $\varphi = \Diamond s_0 \wedge \Diamond s_4$ (visit s_0 and s_4)

Winning strategy for Player 0 from s_0:

- From s_0 to s_3, from s_3 to s_4, and from s_4 to s_1.

Note: this strategy is not winning from s_3 or s_4.

Winning strategy for Player 0 from s_3:

- From s_0 to s_3, from s_4 to s_3, and from s_3 to s_0 on first visit, otherwise to s_4.

Determinacy

Recall: the winning regions are disjoint, i.e., $W_0 \cap W_1 = \emptyset$

Question: Is every state winning for some player?

A game (G, ϕ) with $G = (S, S_0, E)$ is called determined if $W_0 \cup W_1 = S$ holds.

Remarks:

1. We will show that all automata theoretic games we consider here are determined.
2. There are games which are not determined (e.g., concurrent games: even/odd sum, paper-rock-scissors)

Strategy Types

In general, a strategy is a function $f : S^+ \to S$.

(Note that sometimes we might define f only partially.)

1. Computable or recursive strategies: f is computable
2. Finite-state strategies: f is computable with a finite-state automaton meaning that f has bounded information about the past (history).
3. Memoryless or positional strategies: f only depends on the current state of the game (no knowledge about history of play)

Positional Strategies

Given a game (G, ϕ) with $G = (S, S_0, E)$, a strategy $f : S^+ \to S$ is called positional or memoryless if for all words $w, w' \in S^+$ with $w = s_0 \dots s_n$ and $w' = s'_0 \dots s'_m$ such that $s_n = s'_m$, $f(w) = f(w')$ holds.

A positional strategy for Player 0 is representable as

1. a function $f : S_0 \to S$
2. a set of edges containing for every Player-0 state s exactly one edge starting in s (and for every Player-1 state s' all edges starting in s')

Finite-state Strategies

A strategy automaton over a game graph $G = (S, S_0, E)$ is a finite-state machine $A = (M, m_0, \delta, \lambda)$ (Mealy machine) with input and output alphabet S, where

- M is a finite set of states (called memory),
- $m_0 \in M$ is an initial state (the initial memory content),
- $\delta : M \times S \to M$ is a transition function (the memory update fct),
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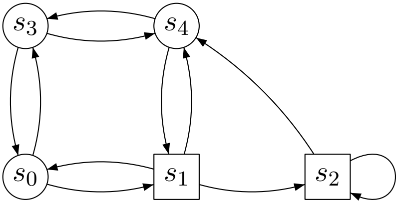
The strategy for Player 0 computed by A is the function

$$f_A(s_0 \dots s_k) := \lambda(\delta(m_0, s_0 \dots s_{k-1}), s_k) \text{ with } s_k \in S_0$$

and the usual extension of δ to words: $\delta(m_0, \epsilon) = m_0$ and $\delta(m_0, s_0 ... s_k) = \delta(\delta(m_0, s_0 ... s_{k-1}), s_k)$. Any strategy f, such that there exists an A with $f_A = f$, is called finite-state strategy.

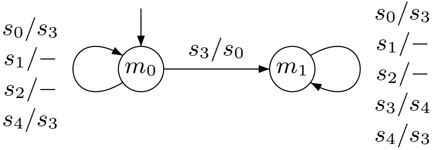
Recall Example



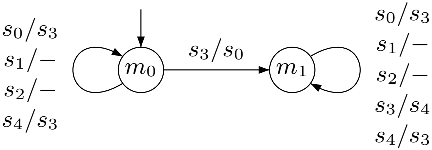
Objective: visit s_0 and s_4, i.e, $\{s_0, s_4\} \subseteq \mathrm{Occ}(\rho)$

Winning strategy for Player 0 from s_0, s_3 and s_4:

- From s_0 to s_3, from s_4 to s_3, and from s_3 to s_0 on first visit, otherwise to s_4.



Extended Game



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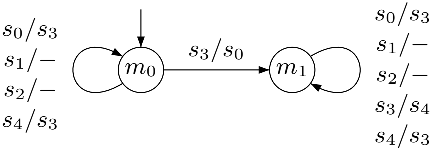

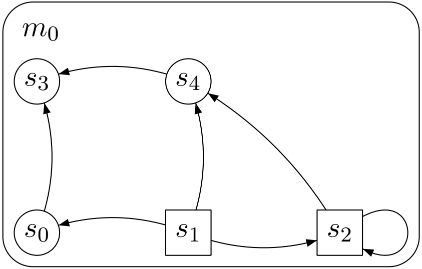

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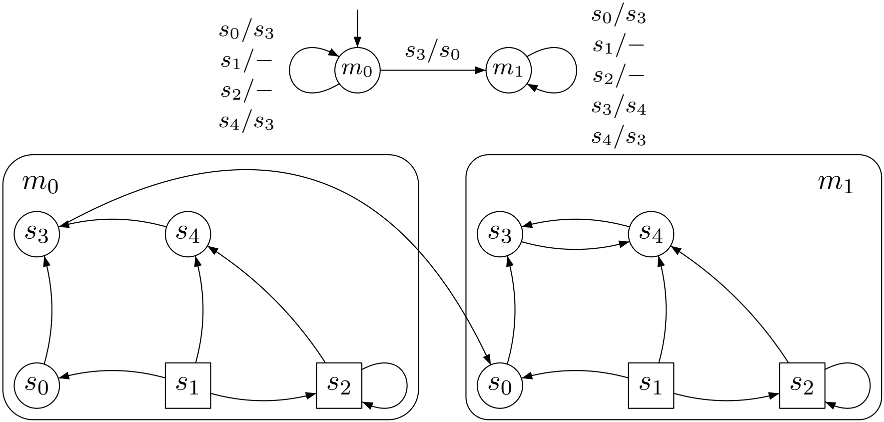
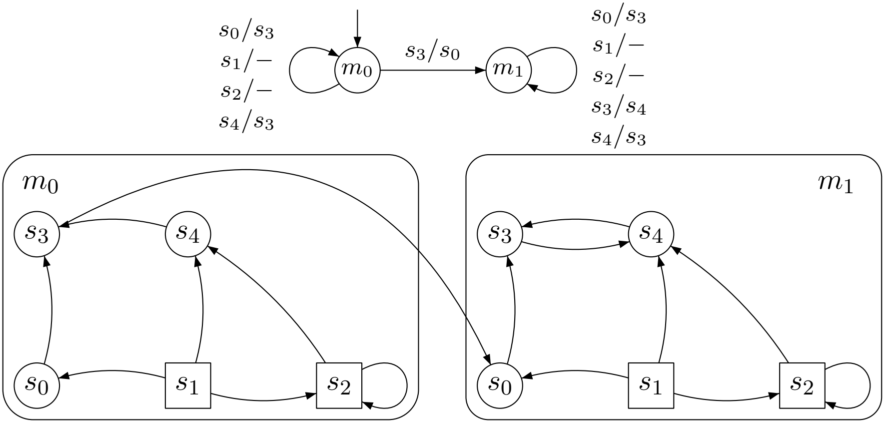


Extended Game



Note: the strategy in the extended grame graph is memoryless.

Reachability and Safety Games

Reachability and Safety Games

Theorem

Given a reachability game (G, F) with $G = (S, S_0, E)$ and $F \subseteq S$, then the winning regions W_0 and W_1 of Player 0 and 1, respectively, are computable, and both players have corresponding memoryless winning strategies.

Proof.

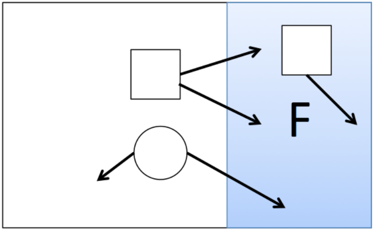
Define

$$\begin{aligned}\mathrm{Attr}_0^i(F) := \{s \in S \mid \quad & \text{Player 0 can force a visit from } s \text{ to } F \\ & \text{in less than } i \text{ moves}\}\end{aligned}$$

Force Visit in Next Step

Given a set of states, compute the set of states $\mathrm{ForceNext}_0(F)$ from which of Player 0 can force to visit F in the next step. I.e., for each state $s \in \mathrm{ForceNext}_0(F)$ Player 0 can fix a strategy s.t. all plays starting in s visit F in the first step.

$$\begin{aligned}\mathrm{ForceNext}_0(F) &= \{s \in S_0 \mid \exists s' \in S : (s, s') \in E \wedge s' \in F\} \cup \\ &\quad \{s \in S_1 \mid \forall s' \in S : (s, s') \in E \rightarrow s' \in F\}\end{aligned}$$

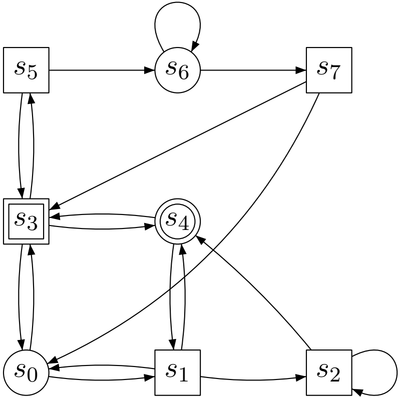


Computing the Attractor

Construction of $\mathrm{Attr}_0^i(F)$:

$$\begin{aligned}
\mathrm{Attr}_0^0(F) &= F \\
\mathrm{Attr}_0^{i+1}(F) &= \mathrm{Attr}_0^i(F) \cup \mathrm{ForceNext}_0(\mathrm{Attr}_0^i(F))
\end{aligned}$$

Example



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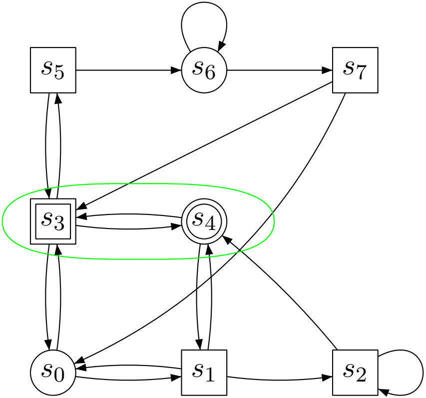
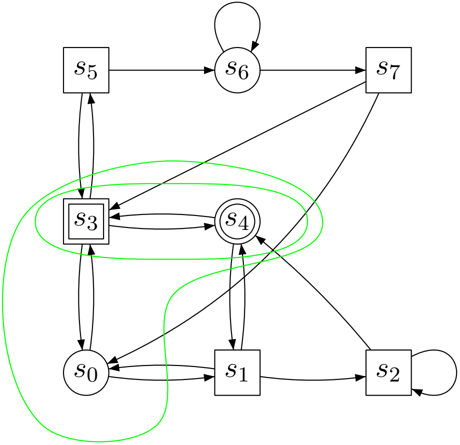


$$\mathrm{Attr}_0^0 = \{s_3, s_4\}$$

Example



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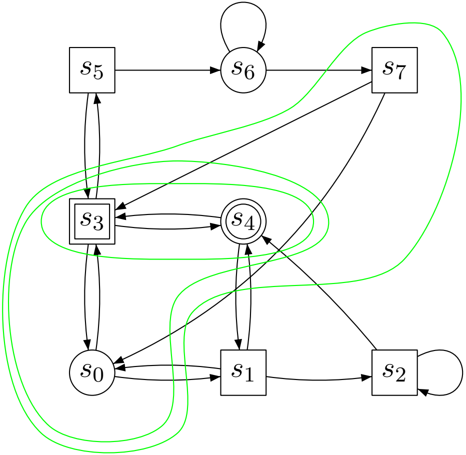

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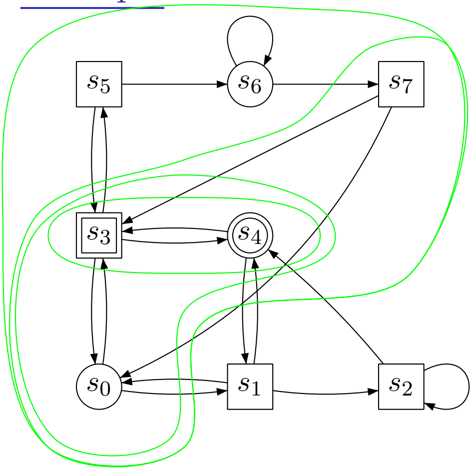
$$\mathrm{Attr}_0^2 = \{s_0, s_3, s_4, s_7\}$$

Example



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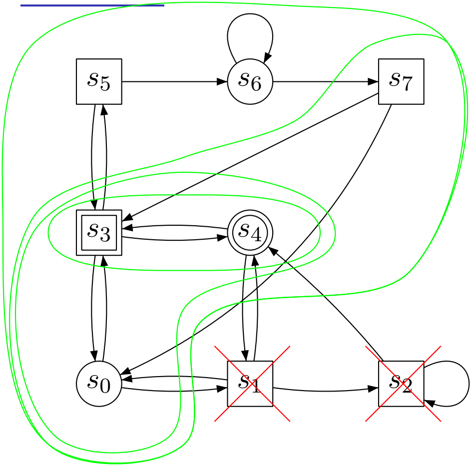
$$\mathrm{Attr}_0^3 = \{s_0, s_3, s_4, s_6, s_7\}$$
$$\mathrm{Attr}_0^4 = \{s_0, s_3, s_4, s_5, s_6, s_7\}$$

Example



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Computing the Attractor

Construction of $\mathrm{Attr}_0^i(F)$:

$$\begin{array}{rcl} \mathrm{Attr}_0^0(F) & = & F \\ \mathrm{Attr}_0^{i+1}(F) & = & \mathrm{Attr}_0^i(F) \cup \mathrm{ForceNext}_0(\mathrm{Attr}_0^i(F)) \end{array}$$

Then $\mathrm{Attr}_0^0(F) \subseteq \mathrm{Attr}_0^1(F) \subseteq \mathrm{Attr}_0^2(F) \subseteq \ldots$ and since S is finite, there exists $k \leq |S|$ s.t. $\mathrm{Attr}_0^k(F) = \mathrm{Attr}_0^{k+1}(F)$.

The 0-Attractor is defined as:

$$\mathrm{Attr}_0(F) := \bigcup_{i=0}^{k} \mathrm{Attr}_0^i(F)$$

Computing the Attractor

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$$\mathrm{Attr}_0(F) := \bigcup_{i=0}^{k} \mathrm{Attr}_0^i(F)$$

Claim: $W_0 = \mathrm{Attr}_0(F)$ and $W_1 = S \setminus \mathrm{Attr}_0(F)$

Duality Between Players

Assume we have a partition of the state space $S = P_0 \cup P_1$ (i.e., $P_0 \cap P_1 = \emptyset$) and we want to prove $W_0 = P_0$ and $W_1 = P_1$.

We want to prove $P_0 \supseteq W_0$, $P_0 \subseteq W_0$, $P_1 \supseteq W_1$, and $P_1 \subseteq W_1$.

Since we know that $W_0 \cap W_1 = \emptyset$ holds, it is sufficient to prove $P_0 \subseteq W_0$ and $P_1 \subseteq W_1$.

$$\begin{aligned} P_0 &\subseteq W_0 \\ S \setminus P_0 &\supseteq S \setminus W_0 \\ P_1 &\supseteq S \setminus W_0 \supseteq W_1 \\ P_1 &\supseteq W_1 \end{aligned} \qquad\qquad \begin{aligned} P_1 &\subseteq W_1 \\ S \setminus P_1 &\supseteq S \setminus W_1 \\ P_0 &\supseteq S \setminus W_1 \supseteq W_0 \\ P_0 &\supseteq W_0 \end{aligned}$$

0-Attractor

To show $W_0 = \mathrm{Attr}_0(F)$ and $W_1 = S \setminus \mathrm{Attr}_0(F)$, we construct winning strategies for Player 0 and 1.

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Proof.

$\text{Attr}_0(F) \subseteq W_0$

We prove for every i and for every state $s \in \text{Attr}_0^i(F)$ that Player 0 has a positional winning strategy to reach F in $\leq i$ steps.

- (Base) $s \in \text{Attr}_0^0(F) = F$
- (Induction) $s \in \text{Attr}_0^{i+1}(F)$
  If $s \in \text{Attr}_0^i(F)$, then we apply induction hypothesis.
  Otherwise $s \in \text{ForceNext}_0(\text{Attr}_0^i(F)) \setminus \text{Attr}_0^i(F)$ and Player 0 can force a visit to $\text{Attr}_0^i(F)$ in one step and from there she needs at move i steps by induction hypothesis. So, F is reached after a finite number of moves.

0-Attractor cont.

Proof cont.

$S \setminus \mathrm{Attr}_0(F) \subseteq W_1$

Assume $s \in S \setminus \mathrm{Attr}_0(F)$, then $s \notin \mathrm{ForceNext}_0(\mathrm{Attr}_0(F))$ and we have two cases:

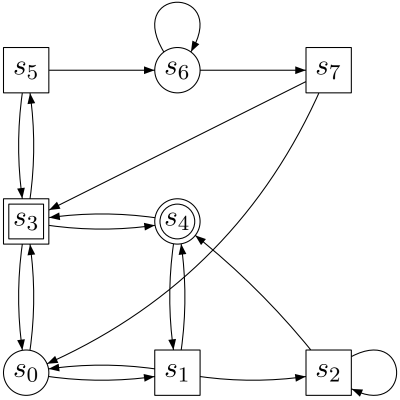
(a) $s \in S_0 \cap (S \setminus \mathrm{Attr}_0(F))$: $\forall s' \in S \,.\, (s, s') \in E \rightarrow s' \notin \mathrm{Attr}_0(F)$

(b) $s \in S_1 \cap (S \setminus \mathrm{Attr}_0(F))$: $\exists s' \in S \,.\, (s, s') \in E \wedge s' \notin \mathrm{Attr}_0(F)$

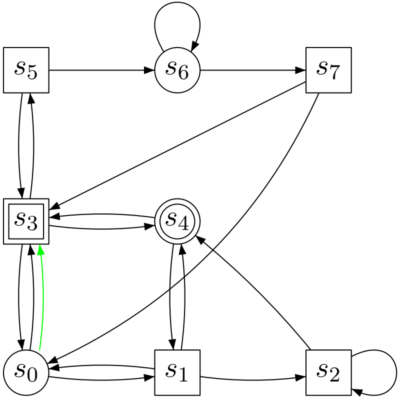
In $S \setminus \mathrm{Attr}_0(F)$ Player 1 can choose edges according to (b) leading again to $S \setminus \mathrm{Attr}_0(F)$ and by (a) Player 0 cannot escape from $S \setminus \mathrm{Attr}_0(F)$. So, F will be avoided forever.

$$W_0 = \mathrm{Attr}_0(F) \text{ and } W_1 = S \setminus \mathrm{Attr}_0(F)$$

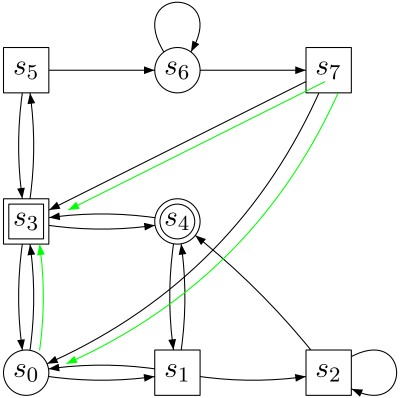
Example of Attractor Strategy



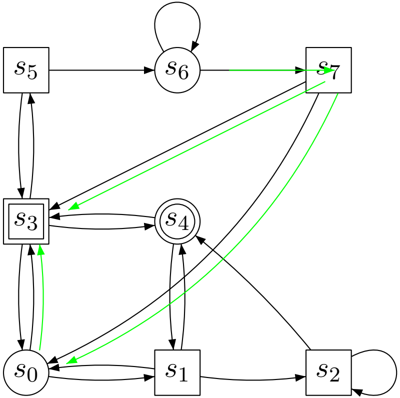
Example of Attractor Strategy



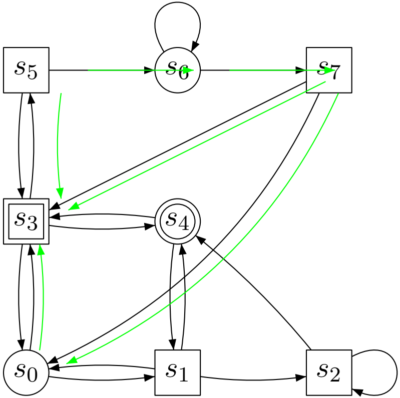
Example of Attractor Strategy



Example of Attractor Strategy



Example of Attractor Strategy



Safety Games

Given a safety game (G, F) with $G = (S, S_0, E)$, i.e.,

$$\phi_S = \{\rho \in S^\omega \mid \mathrm{Occ}(\rho) \subseteq F\},$$

consider the reachability game $(G, S \setminus F)$, i.e.,

$$\phi_R = \{\rho \in S^\omega \mid \mathrm{Occ}(\rho) \cap (S \setminus F) \neq \emptyset\}.$$

$$\begin{aligned} \text{Then, } S^\omega \setminus \phi_R &= \{\rho \in S^\omega \mid \mathrm{Occ}(\rho) \cap (S \setminus F) = \emptyset\} \\ &= \{\rho \in S^\omega \mid \mathrm{Occ}(\rho) \subseteq F\}. \end{aligned}$$

Player 0 has a safety objective in (G, F) $\iff$

Player 1 has a reachability objective in $(G, S \setminus F)$.

So, W_0 in the safety game (G, F) corresponds to W_1 in the reachability game $(G, S \setminus F)$.

Summary

We know how to solve reachability and safety games by positional winning strategies.

The strategies are

- Player 0: Decrease distance to F
- Player 1: Stay outside of $\mathrm{Attr}_0(F)$

In LTL, $\Diamond F$ = reachability and $\Box F$ = safety.

Next, $\Box\Diamond F$ = Büchi and $\Diamond\Box F$ = Co-Büchi.

Hierarchy

4. Reactivity: Muller, Parity

3. Recurrence: Büchi — Persistence: co-Büchi

2. Obligation: Staiger-Wagner, Weak-Parity

1. Reachability — Safety

Büchi and co-Büchi Games

Büchi Game

Given a Büchi game (G, F) over the game graph $G = (S, S_0, E)$ with the set $F \subseteq S$ of Büchi states, we aim to

- determine the winning regions of Player 0 and 1
- compute their respective winning strategies

Recall, Player 0 wins ρ iff she visits infinitely often states in F, i.e., $\phi = \{\rho \in S^\omega \mid \inf(\rho) \cap F \neq \emptyset\}$.

Idea

Compute for $i \geq 1$ the set Recur_0^i of accepting states $s \in F$ from which Player 0 can force at least i revisits to F.

Then, we will show that

$$F \supseteq \mathrm{Recur}_0^1(F) \supseteq \mathrm{Recur}_0^2(F) \supseteq \dots$$

and we compute the winning region of Player 0 with

$$\mathrm{Recur}_0(F) := \bigcap_{i \leq 1} \mathrm{Recur}_0^i(F)$$

Since F is finite, there exists k such that $\mathrm{Recur}_0(F) = \mathrm{Recur}_0^k(F)$.

One-Step Attractor

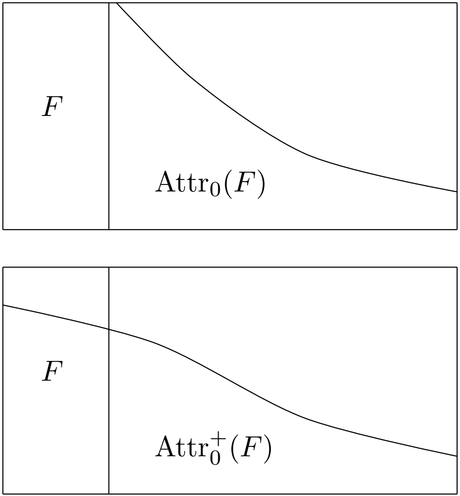
First, we define Recur_0 formally using a modified version of Attractor. We count revisits, so we need the set of states from which Player 0 can force a revisit to F, i.e., state from which she can force a visit in ≥ 1 steps.

We define a slightly modified attractor:

$$\begin{aligned} A_0^0 &= \emptyset \\ A_0^{i+1} &= A_0^i \cup \mathrm{ForceNext}_0(A_0^i \cup F) \end{aligned}$$

$$\mathrm{Attr}_0^+(F) = \bigcup_{i \geq 0} A_0^i$$

Visit versus Revisit



Recurrence Set

We define

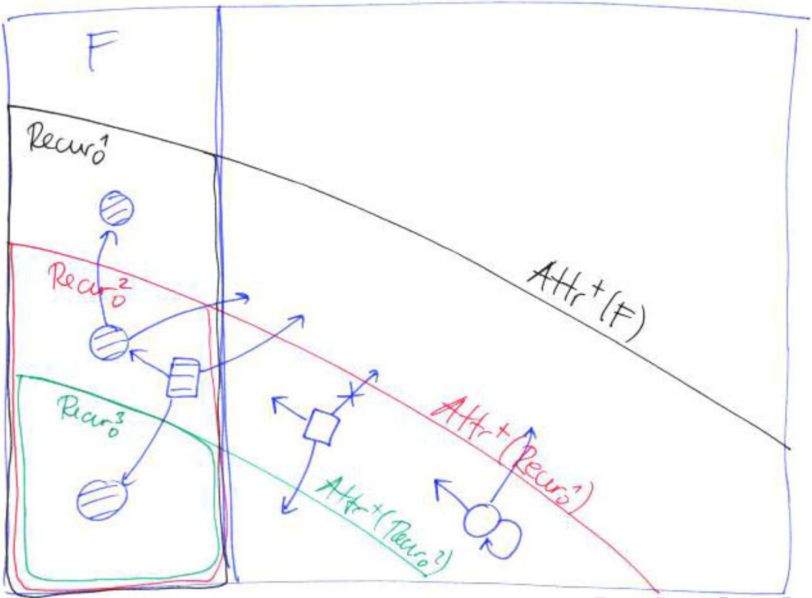
$$\mathrm{Recur}_0^0(F) := F$$
$$\mathrm{Recur}_0^{i+1}(F) := F \cap \mathrm{Attr}_0^+(\mathrm{Recur}_0^i(F))$$
$$\mathrm{Recur}_0(F) := \bigcap_{i \geq 0} \mathrm{Recur}_0^i(F)$$

We show that there exists k such that $\mathrm{Recur}_0(F) := \bigcap_{i \geq 0}^k \mathrm{Recur}_0^i(F)$ by proving $\mathrm{Recur}_0^{i+1}(F) \subseteq \mathrm{Recur}_0^i(F)$ for all $i \geq 0$.

Proof.

- $i = 0$: $F \cap \mathrm{Attr}_0^+(F) \subseteq F$
- $i \to i+1$:
  $\mathrm{Recur}_0^{i+1}(F) = F \cap \mathrm{Attr}_0^+(\mathrm{Recur}_0^i(F)) \subseteq F \cap \mathrm{Attr}_0^+(\mathrm{Recur}_0^{i-1}(F))$ $= \mathrm{Recur}_0^i(F)$ since (i) $\mathrm{Recur}_0^i(F) \subseteq \mathrm{Recur}_0^{i-1}(F)$ by ind. hyp. and (ii) Attr_0^+ is monotone.

Recurrence Set cont.



Recurrence Set cont.

We show that all states in $\mathrm{Attr}_0(\mathrm{Recur}_0(F))$ are winning for Player 0, i.e., $\mathrm{Attr}_0(\mathrm{Recur}_0(F)) \subseteq W_0$. We construct a memoryless winning strategy for Player 0 for all states in $\mathrm{Attr}_0(\mathrm{Recur}_0(F))$.

Proof.

We know that there exists k such that
$\mathrm{Recur}_0^{k+1}(F) = \mathrm{Recur}_0^k(F) = F \cap \mathrm{Attr}_0^+(\mathrm{Recur}_0^k(F))$. So,

- for $s \in \mathrm{Recur}_0^k(F) \cap S_0$ Player 0 can choose an edge back to $\mathrm{Attr}_0^+(\mathrm{Recur}_0^k(F))$ and
- for $s \in \mathrm{Recur}_0^k(F) \cap S_1$ all edges lead back to $\mathrm{Attr}_0^+(\mathrm{Recur}_0^k(F))$.

For all states in $\mathrm{Attr}_0(\mathrm{Recur}_0(F)) \setminus \mathrm{Recur}_0(F)$, Player 0 can follow the attractor strategy to reach $\mathrm{Recur}_0(F)$.

$\square$

Recurrence Set cont.

We show $S \setminus \mathrm{Attr}_0(\mathrm{Recur}_0(F)) \subseteq W_1$.

Proof.

We know that there exists k such that $\mathrm{Recur}_0(F) = \mathrm{Recur}_0^k(F)$, i.e., $S \setminus \mathrm{Attr}_0(\mathrm{Recur}_0(F)) = S \setminus \mathrm{Attr}_0(\mathrm{Recur}_0^k(F))$.

Show: Player 1 can force $\leq i$ visits to F from $s \notin \mathrm{Attr}_0(\mathrm{Recur}_0^i(F))$

$i = 0$: $s \notin \mathrm{Attr}_0(F)$, so Player 1 can avoid visiting F at all.

$i \to i+1$: $s \notin \mathrm{Attr}_0(\mathrm{Recur}_0^{i+1}(F))$.

- $s \notin \mathrm{Attr}_0(\mathrm{Recur}_0^i(F))$, Player 1 plays according to ind. hypothese
- Otherwise, $s \in \mathrm{Attr}_0(\mathrm{Recur}_0^i(F)) \setminus \mathrm{Attr}_0(\mathrm{Recur}_0^{i+1}(F))$ and Player 1 can avoid $\mathrm{Attr}_0(\mathrm{Recur}_0^{i+1}(F))$.

$\square$

Büchi games

We have shown that Player 0 has a (memoryless) winning strategy from every state in $\mathrm{Attr}_0(\mathrm{Recur}_0(F))$, so $\mathrm{Attr}_0(\mathrm{Recur}_0(F)) \subseteq W_0$. And, Player 1 has a (memoryless) winning strategy from every state in $S \setminus \mathrm{Attr}_0(\mathrm{Recur}_0(F))$, so $S \setminus \mathrm{Attr}_0(\mathrm{Recur}_0(F)) \subseteq W_1$. This implies the following theorem.

Theorem

Given a Büchi game $((S, S_0, E), F)$, the winning regions W_0 and W_1 are computable and form a partition, i.e., $W_0 \cup W_1 = S$. Both players have memoryless winning strategies.

Co-Büchi Games

Given a Co-Büchi Game $((S, S_0, E), F)$, i.e.,

$$\phi_C = \{\rho \in S^\omega \mid \mathrm{Inf}(\rho) \subseteq F\}$$

consider the Büchi Game $((S, S_0, E), S \setminus F)$, i.e,

$$\phi_B = \{\rho \in S^\omega \mid \mathrm{Inf}(\rho) \cap (S \setminus F) \neq \emptyset\}.$$

$$\begin{aligned} \text{Then, } S^\omega \setminus \phi_B &= \{\rho \in S^\omega \mid \mathrm{Inf}(\rho) \cap (S \setminus F) = \emptyset\} \\ &= \{\rho \in S^\omega \mid \mathrm{Inf}(\rho) \subseteq F\}. \end{aligned}$$

Player 0 has a co-Büchi objective in (G, F) $\iff$

Player 1 has a Büchi objective in $(G, S \setminus F)$.

So, W_0 in the co-Büchi game (G, F) corresponds to W_1 in the Büchi game $(G, S \setminus F)$.

Summary

We know how to solve Büchi and Co-Büchi games by positional winning strategies.

In LTL,

- $\Diamond F$ = reachability
- $\Box F$ = safety
- $\Box\Diamond F$ = Büchi
- $\Diamond\Box F$ = Co-Büchi

Exercise

2. Consider the game graph shown in below and the following winning conditions:
   (a) $\mathrm{Occ}(\rho) \cap \{1\} \neq \emptyset$ and
   (b) $\mathrm{Occ}(\rho) \subseteq \{1, 2, 3, 4, 5, 6\}$ and
   (c) $\mathrm{Inf}(\rho) \cap \{4, 5\} \neq \emptyset$.

Compute the winning regions and corresponding winning strategies showing the intermediate steps (i.e., the Attractor and Recurrence sets) of the computation.