Automata on Finite Words

Definition

A *nondeterministic finite automaton* (NFA) over Σ is a 4-tuple $A = \langle S, I, T, F \rangle$, where:

- S is a finite set of *states*,

- $I \subseteq S$ is a set of *initial states*,

- $T \subseteq S \times \Sigma \times S$ is a *transition relation*,

- $F \subseteq S$ is a set of *final states*.

We denote $T(s, \alpha) = \{s' \in S \mid (s, \alpha, s') \in T\}$. When T is clear from the context we denote $(s, \alpha, s') \in T$ by $s \xrightarrow{\alpha} s'$.

Runs and Acceptance Conditions

Given a finite word $w \in \Sigma^*$, $w = \alpha_1 \alpha_2 \ldots \alpha_n$, a *run* of A over w is a finite sequence of states $s_1, s_2, \ldots, s_n, s_{n+1}$ such that $s_1 \in I$ and $s_i \xrightarrow{\alpha_i} s_{i+1}$ for all $1 \leq i \leq n$.

A run over w between s_i and s_j is denoted as $s_i \xrightarrow{w} s_j$.

The run is said to be *accepting* if and only if $s_{n+1} \in F$. If A has an accepting run over w, then we say that A *accepts* w.

The language of A, denoted $\mathcal{L}(A)$ is the set of all words accepted by A.

A set of words $S \subseteq \Sigma^*$ is *recognizable* if there exists an automaton A such that $S = \mathcal{L}(A)$.

Determinism and Completeness

Definition 1 *An automaton $A = \langle S, I, T, F \rangle$ is* deterministic *(DFA) if and only if $\|I\| \leq 1$ and, for each $s \in S$ and for each $\alpha \in \Sigma$, $\|T(s, \alpha)\| \leq 1$.*

If A is deterministic we write $T(s, \alpha) = s'$ instead of $T(s, \alpha) = \{s'\}$.

Definition 2 *An automaton $A = \langle S, I, T, F \rangle$ is* complete *if and only if $\|I\| \geq 1$ and, for each $s \in S$ and for each $\alpha \in \Sigma$, $\|T(s, \alpha)\| \geq 1$.*

Determinism and Completeness

Proposition 1 *If A is deterministic, then it has at most one run for each input word.*

Proposition 2 *If A is complete, then it has at least one run for each input word.*

Determinization

Theorem 1 *For every NFA A there exists a DFA A_d such that $\mathcal{L}(A) = \mathcal{L}(A_d)$.*

Let $A_d = \langle 2^S, \{I\}, T_d, \{G \subseteq S \mid G \cap F \neq \emptyset\}\rangle$, where

$$(S_1, \alpha, S_2) \in T_d \iff S_2 = \{s' \mid \exists s \in S_1 \; . \; (s, \alpha, s') \in T\}$$

This definition is known as subset construction.

Exercise 1 *Let $\Sigma = \{a, b\}$ and $L_n = \{uav \mid u, v \in \Sigma^*, \; |v| = n - 1\}$, for each integer $n \geq 1$. Build an NFA that recognizes L_n and apply subset construction to it.*

Completion

Lemma 1 *For every NFA A there exists a complete NFA A_c such that $\mathcal{L}(A) = \mathcal{L}(A_c)$.*

Let $A_c = \langle S \cup \{\sigma\}, I, T_c, F \rangle$, where $\sigma \notin S$ is a new sink state. The transition relation T_c is defined as:

$$\forall s \in S \forall \alpha \in \Sigma \ . \ (s, \alpha, \sigma) \in T_c \iff \forall s' \in S \ . \ (s, \alpha, s') \notin T$$

and $\forall \alpha \in \Sigma \ . \ (\sigma, \alpha, \sigma) \in T_c$.

Remark: The subset construction yields a complete deterministic automaton, with sink state $\emptyset$.

Closure Properties

Theorem 2 *Let $A_1 = \langle S_1, I_1, T_1, F_1 \rangle$ and $A_2 = \langle S_2, I_2, T_2, F_2 \rangle$ be two NFA, such that $S_1 \cap S_2 = \emptyset$. There exists automata $\bar{A}_1$, $A_\cup$ and $A_\cap$ that recognize the languages $\Sigma^* \setminus \mathcal{L}(A_1)$, $\mathcal{L}(A_1) \cup \mathcal{L}(A_2)$, and $\mathcal{L}(A_1) \cap \mathcal{L}(A_2)$, respectivelly.*

Let $A' = \langle S', I', T', F' \rangle$ be the complete and deterministic (why?) automaton such that $\mathcal{L}(A_1) = \mathcal{L}(A')$, and $\bar{A}_1 = \langle S', I', T', S' \setminus F' \rangle$.

Let $A_\cup = \langle S_1 \cup S_2, I_1 \cup I_2, T_1 \cup T_2, F_1 \cup F_2 \rangle$.

Let $A_\cap = \langle S_1 \times S_2, I_1 \times I_2, T_\cap, F_1 \times F_2 \rangle$ where:

$$(\langle s_1, t_1 \rangle, \alpha, \langle s_2, t_2 \rangle) \in T_\cap \iff (s_1, \alpha, s_2) \in T_1 \text{ and } (t_1, \alpha, t_2) \in T_2$$

On the Exponential Blowup of Complementation

Theorem 3 *For every $n \in \mathbb{N}$, $n \geq 1$, there exists an automaton A, with $size(A) = n + 1$ such that no deterministic automaton with less than 2^n states recognizes the complement of $\mathcal{L}(A)$.*

Let $\Sigma = \{a, b\}$ and $L_n = \{uav \mid u, v \in \Sigma^*, |v| = n - 1\}$, for all $n \geq 1$.

There exists a NFA with exactly $n + 1$ states which recognizes L_n.

Suppose that $B = \langle S, \{s_0\}, T, F \rangle$, is a (complete) DFA with $\|S\| < 2^n$ that accepts $\Sigma^* \setminus L_n$.

On the Exponential Blowup of Complementation

$\|\{w \in \Sigma^* \mid |w| = n\}\| = 2^n$ and $\|S\| < 2^n$ (by the pigeonhole principle)

$\Rightarrow \exists uav_1, ubv_2 \ . \ |uav_1| = |ubv_2| = n$ and $s \in S \ . \ s_0 \xrightarrow{uav_1} s$ and $s_0 \xrightarrow{ubv_2} s$

Let s_1 be the (unique) state of B such that $s \xrightarrow{u} s_1$.

Since $|uav_1| = n$, then $uav_1u \in L_n \Rightarrow uav_1u \notin \mathcal{L}(B)$, i.e. s is not accepting.

On the other hand, $ubv_2u \notin L_n \Rightarrow ubv_2u \in \mathcal{L}(B)$, i.e. s is accepting, contradiction.

Projections

Let the input alphabet be $\Sigma = \Sigma_1 \times \Sigma_2$. Any word $w \in \Sigma^*$ can be uniquely identified to a pair $\langle w_1, w_2 \rangle \in \Sigma_1^* \times \Sigma_2^*$ such that $|w_1| = |w_2| = |w|$.

The *projection* operations are
$$pr_1(L) = \{v \in \Sigma_2^* \mid \langle u, v \rangle \in L, \text{ for some } u \in \Sigma_1^*\} \text{ and}$$
$$pr_2(L) = \{u \in \Sigma_1^* \mid \langle u, v \rangle \in L, \text{ for some } v \in \Sigma_2^*\}.$$

Theorem 4 *If the language $L \subseteq (\Sigma_1 \times \Sigma_2)^*$ is recognizable, then so are the projections $pr_i(L)$, for $i = 1, 2$.*

Remark

The operations of union, intersection and complement correspond to the boolean $\vee$, $\wedge$ and $\neg$.

The projection corresponds to the first-order existential quantifier $\exists x$.

The Myhill-Nerode Theorem

Let $A = \langle S, I, T, F \rangle$ be an automaton over the alphabet Σ^*.

Define the relation $\sim_A \subseteq \Sigma^* \times \Sigma^*$ as:

$$u \sim_A v \iff [\forall s, s' \in S \;.\; s \xrightarrow{u} s' \iff s \xrightarrow{v} s']$$

$\sim_A$ is an equivalence relation of finite index

Let $L \subseteq \Sigma^*$ be a language. Define the relation $\sim_L \subseteq \Sigma^* \times \Sigma^*$ as:

$$u \sim_L v \iff [\forall w \in \Sigma^* \;.\; uw \in L \iff vw \in L]$$

$\sim_L$ is an equivalence relation

The Myhill-Nerode Theorem

Theorem 5 *A language $L \subseteq \Sigma^*$ is recognizable iff $\sim_L$ is of finite index.*

"$\Rightarrow$" Suppose $L = \mathcal{L}(A)$ for some automaton A.

$\sim_A$ is of finite index.

for all $u, v \in \Sigma^*$ we have $u \sim_A v \Rightarrow u \sim_L v$

index of $\sim_L \leq$ index of $\sim_A < \infty$

The Myhill-Nerode Theorem

"$\Longleftarrow$" $\sim_L$ is an equivalence relation of finite index, and let $[u]$ denote the equivalence class of $u \in \Sigma^*$.

$A = \langle S, I, T, F \rangle$, where:

- $S = \{[u] \mid u \in \Sigma^*\}$,

- $I = [\epsilon]$,

- $[u] \xrightarrow{\alpha} [v] \iff u\alpha \sim_L v$,

- $F = \{[u] \mid u \in L\}$.

For DFA all minimal automata are isomorphic.

For NFA there may be more non-isomorphic minimal automata.

Pumping Lemma

Lemma 2 (Pumping) *Let $A = \langle S, I, T, F \rangle$ be a finite automaton with $size(A) = n$, and $w \in \mathcal{L}(A)$ be a word of length $|w| \geq n$. Then there exists three words $u, v, t \in \Sigma^*$ such that:*

1. *$|v| \geq 1$,*

2. *$w = uvt$ and,*

3. *for all $k \geq 0$, $uv^k t \in \mathcal{L}(A)$.*

Example

$L = \{a^n b^n \mid n \in \mathbb{N}\}$ is not recognizable:

Suppose that there exists an automaton A with $size(A) = N$, such that $L = \mathcal{L}(A)$.

Consider the word $a^n b^n \in L = \mathcal{L}(A)$, such that $n \geq N$.

There exists words u, v, w such that $|v| \geq 1$, $uvw = a^n b^n$ and $uv^k w \in L$ for all $k \geq 1$. Then $a^{n+k|v|} b^n \in L$, for all $k \geq 1$, contradiction.

Decidability

Given nondeterministic finite automata A and B:

- **Emptiness** $\mathcal{L}(A) = \emptyset$?

- **Inclusion** $\mathcal{L}(A) \subseteq \mathcal{L}(B)$?

- **Equivalence** $\mathcal{L}(A) = \mathcal{L}(B)$?

- **Infinity** $\|\mathcal{L}(A)\| < \infty$?

- **Universality** $\mathcal{L}(A) = \Sigma^*$?

Emptiness

Theorem 6 *Let A be an automaton with $size(A) = n$. If $\mathcal{L}(A) \neq \emptyset$, then there exists a word of length less than n that is accepted by A.*

Let u be the shortest word in $\mathcal{L}(A)$.

If $|u| < n$ we are done.

If $|u| \geq n$, there exists $u_1, v, u_2 \in \Sigma^*$ such that $|v| > 1$ and $u_1 v u_2 = u$.

Then $u_1 u_2 \in \mathcal{L}(A)$ and $|u_1 u_2| < |u_1 v u_2|$, contradiction.

Everything is decidable

Theorem 7 *The emptiness, equality, infinity and universality problems are decidable for automata on finite words.*

Although complexity varies from problem to problem:

- **Emptiness** ($\mathcal{L}(A) = \emptyset$) belongs to NLOGSPACE

- **Inclusion** ($\mathcal{L}(A) \subseteq \mathcal{L}(B)$) is PSPACE-complete

- **Equivalence** ($\mathcal{L}(A) = \mathcal{L}(B)$) is PSPACE-complete

- **Infinity** ($\|\mathcal{L}(A)\| < \infty$) belongs to NLOGSPACE

- **Universality** ($\mathcal{L}(A) = \Sigma^*$) is PSPACE-complete

Automata on Finite Words and WS1S

WS1S

Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet.

Any finite word $w \in \Sigma^*$ induces the *finite* sets $p_a = \{p \mid w(p) = a\}$.

- $x \leq y :$ x is less than y,

- $s(x) = y :$ y is the successor of x,

- $p_a(x) :$ a occurs at position x in w

Remember that $\leq$ and $s(.)$ can be defined one from another.

Problem Statement

Given a sentence φ in WS1S, let $\mathcal{L}(\varphi) = \{w \mid \mathfrak{m}_w \models \varphi\}$, where $\mathfrak{m}_w = \langle dom(w), \{\bar{p}_a\}_{a \in \Sigma}, \leq \rangle$, such that:

- $dom(w) = \{0, 1, \ldots, n-1\}$,

- $\bar{p}_a = \{x \in dom(w) \mid w(x) = a\}$,

A language $L \subseteq \Sigma^*$ is said to be WS1S-*definable* iff there exists a WS1S sentence φ such that $L = \mathcal{L}(\varphi)$.

1. Given A build φ_A such that $\mathcal{L}(A) = \mathcal{L}(\varphi)$

2. Given φ build A_φ such that $\mathcal{L}(A) = \mathcal{L}(\varphi)$

The recognizable and WS1S-definable languages coincide

Coding of Σ

Let $m \in \mathbb{N}$ be the smallest number such that $\|\Sigma\| \leq 2^m$.

W.l.o.g. assume that $\Sigma = \{0,1\}^m$, and let $X_1 \ldots X_p, x_{p+1}, \ldots x_m$

A word $w \in \Sigma^*$ induces an *interpretation* of $X_1 \ldots X_p, x_{p+1}, \ldots x_m$:

- $i \in \iota_w(X_j)$ iff the j-th element of w_i is 1, and

- $\iota_w(x_j) = i$ iff w_i has 1 on the j-th position and, for all $k \neq i$ w_k has 0 on the j-th position.

Example

Example 1 *Let $\Sigma = \{a, b, c, d\}$, encoded as $a = (00)$, $b = (01)$, $c = (10)$ and $d = (11)$. Then the word abbaacdd induces the valuation $X_1 = \{5, 6, 7\}$, $X_2 = \{1, 2, 6, 7\}$.* □

From Automata to Formulae

Let $A = \langle S, I, T, F \rangle$ with $S = \{s_1, \ldots, s_p\}$, and $\Sigma = \{0, 1\}^m$.

Build $\Phi_A(X_1, \ldots, X_m)$ such that $\forall w \in \Sigma^* \, . \, w \in \mathcal{L}(A) \iff [\![\Phi_A]\!]_{\iota_w}^{\mathfrak{m}_w} = \text{true}$

Let $a \in \{0, 1\}^m$. Let $\Phi_a(x, X_1, \ldots, X_m)$ be the conjunction of:

- $X_i(x)$ if the $a_i = 1$, and

- $\neg X_i(x)$ otherwise.

For all $w \in \Sigma^*$ we have $\mathfrak{m}_w \models \forall x \, . \, \bigvee_{a \in \Sigma} \Phi_a(x, \vec{X})$

Notice that $\Phi_a \wedge \Phi_b$ is unsatisfiable, for $a \neq b$.

Coding of S

Let $\{Y_1, \ldots, Y_p\}$ be set variables.

Y_i is the set of all positions labeled by A with state s_i during some run

$$\Phi_S(Y_1, \ldots, Y_p) \;:\; \forall z \;.\; \bigvee_{1 \leq i \leq p} Y_i(z) \;\wedge\; \bigwedge_{1 \leq i < j \leq p} \neg \exists z \;.\; Y_i(z) \wedge Y_j(z)$$

Coding of I

Every run starts from an initial state:

$$\Phi_I(Y_1, \ldots, Y_p) \; : \; \exists x \forall y \; . \; x \leq y \wedge \bigvee_{s_i \in I} Y_i(x)$$

Coding of T

Consider the transition $s_i \xrightarrow{a} s_j$:

$$\Phi_T(X_1, \ldots, X_m, Y_1, \ldots, Y_p) \ : \ \forall x . \ \neg last(x) \wedge Y_i(x) \wedge \Phi_a(x, \vec{X}) \to \bigvee_{(s_i, a, s_j) \in T} Y_j(s(x))$$

where $last(x) = \forall y . \ y \leq x$

Coding of F

The last state on the run is a final state:

$$\Phi_F(Y_1, \ldots, Y_p) \; : \; \exists x \,.\, last(x) \wedge \bigvee_{s_i \in F} Y_i(x)$$

$$\Phi_A = \exists Y_1 \ldots \exists Y_p \,.\, \Phi_S \wedge \Phi_I \wedge \Phi_T \wedge \Phi_F$$

From Formulae to Automata

Let $\Phi(X_1, \ldots, X_p, x_{p+1}, \ldots, x_m)$ be a WS1S formula.

Build an automaton A_Φ such that $\forall w \in \Sigma^* \, . \, w \in \mathcal{L}(A) \iff [\![\Phi]\!]_{\iota_w}^{\mathfrak{m}_w} = \text{true}$

Let $\Phi(X_1, X_2, x_3, x_4)$ be:

1. $X_1(x_3)$

2. $x_3 \leq x_4$

3. $X_1 = X_2$

From Formulae to Automata

A_Φ is built by induction on the structure of Φ:

- for $\Phi = \phi_1 \wedge \phi_2$ we have $\mathcal{L}(A_\Phi) = \mathcal{L}(A_{\phi_1}) \cap \mathcal{L}(A_{\phi_2})$

- for $\Phi = \phi_1 \vee \phi_2$ we have $\mathcal{L}(A_\Phi) = \mathcal{L}(A_{\phi_1}) \cup \mathcal{L}(A_{\phi_2})$

- for $\Phi = \neg\phi$ we have $\mathcal{L}(A_\Phi) = \overline{\mathcal{L}(A_\phi)}$

- for $\Phi = \exists X_i \; . \; \phi$, we have $\mathcal{L}(A_\Phi) = pr_i(\mathcal{L}(A_\phi))$.

Consequences

Theorem 8 *A language $L \subseteq \Sigma^*$ is definable in WS1S iff it is recognizable.*

Corollary 1 *The SAT problem for WS1S is decidable.*

Exercise 2 *Prove that there is no WS1S formula $\varphi(x, y, z)$ that defines the relation $\{(m, n, p) \in \mathbb{N}^3 \mid m + n = p\}$.*

Regular, Star Free and Aperiodic Languages

Regular Languages

Let Σ be an alphabet, and $X, Y \subseteq \Sigma^*$

$$XY = \{xy \mid x \in X \text{ and } y \in Y\}$$

$$X^* = \{x_1 \ldots x_n \mid n \geq 0, \ x_1, \ldots, x_n \in X\}$$

The class of *regular languages* $\mathcal{R}(\Sigma)$ is the smallest class of languages $L \subseteq \Sigma^*$ such that:

- $\emptyset, \{\epsilon\} \in \mathcal{R}(\Sigma)$

- $\{\alpha\} \in \mathcal{R}(\Sigma)$, for all $\alpha \in \Sigma$

- if $X, Y \in \mathcal{R}(\Sigma)$ then $X \cup Y, XY, X^* \in \mathcal{R}(\Sigma)$

Regular, rational and recognizable languages

Theorem 9 (Kleene) *A set of finite words is recognizable if and only if it is regular.*

Proof in every textbook.

Rational = regular, in older books e.g.

Samuel Eilenberg. *Automata, Languages and Machines.* Academic Press, 1974

Star Free Languages

The class of *star-free languages* is the smallest class $SF(\Sigma)$ of languages $L \in \Sigma^*$ such that:
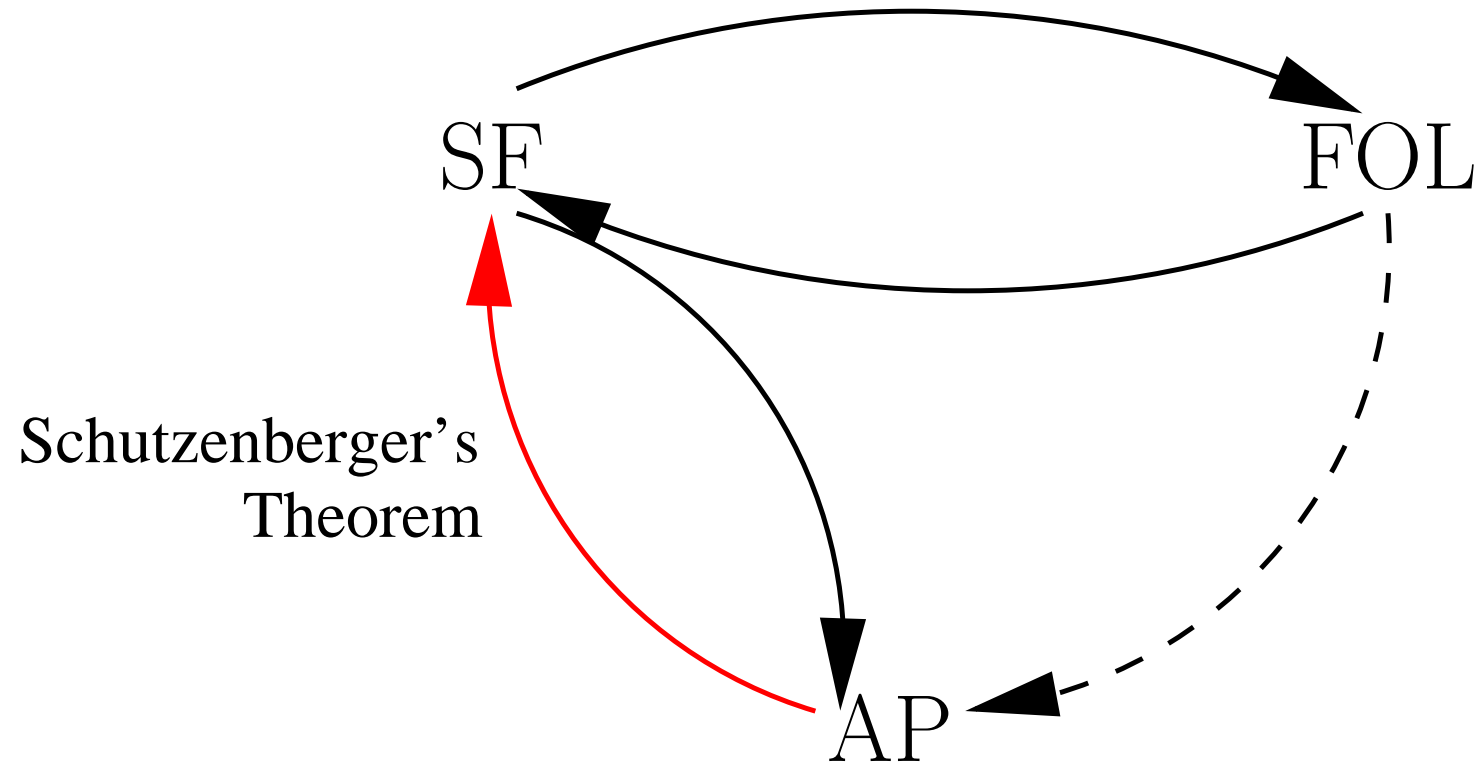
- $\emptyset, \{\epsilon\} \in SF(\Sigma)$ and $\{a\} \in SF(\Sigma)$ for all $a \in \Sigma$

- if $X, Y \in SF(\Sigma)$ then $X \cup Y, XY, \overline{X} \in SF(\Sigma)$

Example 2

- $\Sigma^* = \overline{\overline{\emptyset}}$ *is star-free*

- *if* $B \subset \Sigma$, *then* $\Sigma^* B \Sigma^* = \bigcup_{b \in B} \Sigma^* b \Sigma^*$ *is star-free*

- *if* $B \subset \Sigma$, *then* $B^* = \overline{\Sigma^* \overline{B} \Sigma^*}$ *is star-free*

- *if* $\Sigma = \{a, b\}$, *then* $(ab)^* = \overline{b\Sigma^* \cup \Sigma^* a \cup \Sigma^* aa \Sigma^* \cup \Sigma^* bb \Sigma^*}$ *is star-free*

Exercise 3 *If* $\Sigma = \{a, b, c\}$, *write* $(ab)^*$ *as a star-free language.*

SF   FOL

Schutzenberger's
Theorem

AP

The Splitting Lemma

Lemma 3 *Let $A, B \subseteq \Sigma$ be subalphabets such that $A \cap B = \emptyset$. Then, for each star-free language $L \in SF(\Sigma)$, we have:*

$$L \cap B^* A B^* = \bigcup_{1 \leq i \leq n} K_i a_i L_i$$

where $a_i \in A$ and $K_i, L_i \in SF(B)$, for all $1 \leq i \leq n$.

W.l.o.g. we prove the case $A = \{a\}$ (why?) by induction on L:

- If $L = \{a\}$ then $L \cap B^* A B^* = \{\epsilon\} a \{\epsilon\}$.

- If $L = \{a'\}$, $a' \neq a$, then $L \cap B^* A B^* = \emptyset a \emptyset$.

- If $L = \Sigma^*$ then $L \cap B^* A B^* = B^* A B^*$.

- If $L = L_1 \cup L_2$ then $L \cap B^* A B^* = (L_1 \cap B^* A B^*) \cup (L_2 \cap B^* A B^*)$.

The Splitting Lemma

$$L \cap B^* A B^* = \bigcup_{1 \leq i \leq n} K_i a_i L_i$$

- If $L = L_1 \cdot L_2$ then
  $$L \cap B^* A B^* = (L_1 \cap B^*) \cdot (L_2 \cap B^* A B^*) \cup (L_1 \cap B^* A B^*) \cdot (L_2 \cap B^*).$$

- Else, if $L = \Sigma^* \setminus L'$, by the inductive hypothesis $L' = \bigcup_{1 \leq i \leq n} K_i' a L_i'$. We assume w.l.o.g that $\{K_i'\}_{i=1}^n$ form a patition of B^*:

  - if $K_i' \cap K_j' \neq \emptyset$, rewrite
    $$K_i' a L_i' \cup K_j' a L_j' = (K_i' \setminus K_j') a L_i' \cup (K_j' \setminus K_i') a L_j' \cup (K_i' \cap K_j') a (L_i' \cup L_j')$$

  - if $\bigcup_{i=1}^n K_i' \subsetneq B^*$, add $(B^* \setminus \bigcup_{i=1}^n K_i') a \emptyset$ to $\{K_i' a L_i'\}_{i=1}^n$
    $$(\Sigma^* \setminus L') \cap B^* a B^* = \bigcup_{i=1}^n K_i' a (B^* \setminus L_i')$$

Subword Formulae

Let $w = a_0 a_1 \ldots a_{n-1}$ be a finite word, and $w(i,j) = a_i a_{i+1} \ldots a_{j-1}$ be a subword of w, $0 \leq i < n$ and $0 \leq j \leq n$, $i < j$.

Proposition 3 *For each FOL sentence φ there exists a formula $\varphi[x,y]$ such that, for each $w \in \Sigma^*$ and each $0 \leq i < j \leq |w|$:*

$$\mathfrak{m}_{w(i,j)} \models \varphi \iff [\![\varphi[x,y]]\!]^{\mathfrak{m}_w}_{[x \leftarrow i][y \leftarrow j]} = \mathbf{true}$$

By induction on the structure of φ:

$$
\begin{aligned}
(\neg\varphi)[x,y] &= \neg(\varphi[x,y]) \\
(\varphi \wedge \psi)[x,y] &= (\varphi[x,y]) \wedge (\psi[x,y]) \\
(\exists z.\varphi)[x,y] &= \exists z \,.\, x \leq z \wedge z < y \wedge \varphi[x,y]
\end{aligned}
$$

Star Free Languages are FOL-definable

For each $L \in SF(\Sigma)$, there exists an FOL sentence φ_L such that:

$$L = \{w \in \Sigma^* \mid \mathfrak{m}_w \models \varphi_L\}$$

By induction on the structure of L:

$$\emptyset = \{w \in \Sigma^* \mid \mathfrak{m}_w \models \bot\} \qquad\qquad \{a\} = \{w \in \Sigma^* \mid \mathfrak{m}_u \models p_a(0) \wedge last(0)\}$$

$$X \cup Y = \{w \in \Sigma^* \mid \mathfrak{m}_u \models \varphi_X \vee \varphi_Y\} \qquad \overline{X} = \{w \in \Sigma^* \mid \mathfrak{m}_u \models \neg\varphi_X\}$$

$$X \cdot Y = \{w \in \Sigma^* \mid \mathfrak{m}_u \models \exists y \exists z \;.\; 0 \leq y \leq z \wedge \varphi_X[0, y] \wedge \varphi_Y[y, z] \wedge last(z)\}$$

FOL-definable Languages are Star Free

Let φ be an FOL formula with $FV(\varphi) = V$ and let $\Sigma_V = \Sigma \times \{0,1\}^V$.

Encode each pair (w, ι), with $\iota : V \to [0, |w| - 1]$ as a word $\overline{(w, \iota)} \in \Sigma_V^*$:

$$\overline{(a_0 \ldots a_{k-1}, \iota)} = (a_0, \tau_0) \ldots (a_{k-1}, \tau_{k-1}), \ \tau_i(x) = 1 \iff \iota(x) = i$$

and let $\mathcal{N}_V = \{\overline{(w, \iota)} \mid w \in \Sigma^*, \iota : V \to [0, |w| - 1]\}$.

Let $\Sigma_V^{x=i} = \{(a, \tau) \mid \tau(x) = i\}$, for $i = 0, 1$

$$\mathcal{N}_V = \bigcap_{x \in V} (\Sigma_V^{x=0})^* (\Sigma_V^{x=1})(\Sigma_V^{x=0})^* \in SF(\Sigma_V)$$

FOL-definable Languages are Star Free

$$\llbracket p_a(x) \rrbracket_V \;\;=\;\; \{\overline{(w,\iota)} \in \mathcal{N}_V \mid w = a_0 \ldots a_{k-1},\ a_{\iota(x)} = a\}$$

$$\llbracket x \le y \rrbracket_V \;\;=\;\; \{\overline{(w,\iota)} \in \mathcal{N}_V \mid \iota(x) \le \iota(y)\}$$

$$\llbracket \phi \vee \psi \rrbracket_V \;\;=\;\; \llbracket \phi \rrbracket_V \cup \llbracket \psi \rrbracket_V$$

$$\llbracket \neg \phi \rrbracket_V \;\;=\;\; \mathcal{N}_V \setminus \llbracket \phi \rrbracket_V$$

$$\llbracket \exists x \,.\, \phi \rrbracket_V \;\;=\;\; \{\overline{(w,\iota)} \in \mathcal{N}_V \mid \exists i \in [0, |w| - 1]\,.\, \overline{(w, \iota[x \leftarrow i])} \in \llbracket \phi \rrbracket_{V \cup \{x\}}\}$$

Proposition 4 *If $\varphi \in FOL$ and $FV(\varphi) \subseteq V$, then $\llbracket \varphi \rrbracket_V \in SF(\Sigma_V)$.*

$$\llbracket p_a(x) \rrbracket_V \;\;=\;\; \mathcal{N}_V \cap (\Sigma_V^* \cdot \{(a,\tau) \mid \tau(x) = 1\} \cdot \Sigma_V^*)$$

$$\llbracket x \le y \rrbracket_V \;\;=\;\; \mathcal{N}_V \cap (\Sigma_V^* \cdot \Sigma_V^{x=1} \cdot \Sigma_V^* \cdot \Sigma_V^{y=1} \cdot \Sigma_V^*)$$

FOL-definable Languages are Star Free

Proposition 5 *If $\varphi \in FOL$ and $FV(\varphi) \subseteq V$, then $[\![\varphi]\!]_V \in SF(\Sigma_V)$.*

If $\varphi = \exists x . \phi$, we assume w.l.o.g. that $x \notin V$ (α-conversion)

$$
\begin{aligned}
[\![\phi]\!]_{V \cup \{x\}} &= [\![\phi]\!]_{V \cup \{x\}} \cap (\Sigma_{V \cup \{x\}}^{x=0})^* (\Sigma_{V \cup \{x\}}^{x=1})(\Sigma_{V \cup \{x\}}^{x=0})^* \\
&= \bigcup_{i=1}^n K_i' a_i' L_i' \text{ (Splitting Lemma)}
\end{aligned}
$$

where $K_i', L_i' \in SF(\Sigma_{V \cup \{x\}}^{x=0})$ and $a_i' \in \Sigma_{V \cup \{x\}}^{x=1}$, for all $1 \leq i \leq n$

Let $\pi : \Sigma_{V \cup \{x\}}^{x=0} \to \Sigma_V$ be the bijective (why?) renaming $(a, \tau) \xmapsto{\pi} (a, \tau \!\downarrow_V)$
Let $K_i = \pi(K_i')$, $L_i = \pi(L_i')$ and $a_i = (a, \tau \!\downarrow_V) \iff a_i = (a, \tau)$

$$
[\![\exists x . \phi]\!]_V = \bigcup_{i=1}^n K_i a_i L_i
$$

NB: SF languages are preserved by bijective renamings (why bijective ?)

Aperiodic Languages

Definition 3 *A language $L \subseteq \Sigma^*$ is said to be* aperiodic *iff:*

$$\exists n_0 \forall n \geq n_0 \forall u, v, t \in \Sigma^* \ . \ uv^n t \in L \iff uv^{n+1} t \in L$$

n_0 *is called the* index *of L.*

Example 3 *$0^* 1^*$ is aperiodic. Let $n_0 = 2$. We have three cases:*

1. *$u, v \in 0^*$ and $t \in 0^* 1^*$:*

$$\forall n \geq 2 \ . \ uv^n t \in L$$

2. *$u \in 0^*, v \in 0^+ 1^+$ and $t \in 1^*$:*

$$\forall n \geq 2 \ . \ uv^n t \notin L$$

3. *$u \in 0^* 1^*, v \in 1^*$ and $t \in 1^*$:*

$$\forall n \geq 2 \ . \ uv^n t \in L$$

Periodic Languages

Conversely, a language $L \subseteq \Sigma^*$ is said to be *periodic* iff:

$$\forall n_0 \exists n \geq n_0 \exists u, v, t \in \Sigma^* . (uv^n t \notin L \wedge uv^{n+1} t \in L) \vee (uv^n t \in L \wedge uv^{n+1} t \notin L)$$
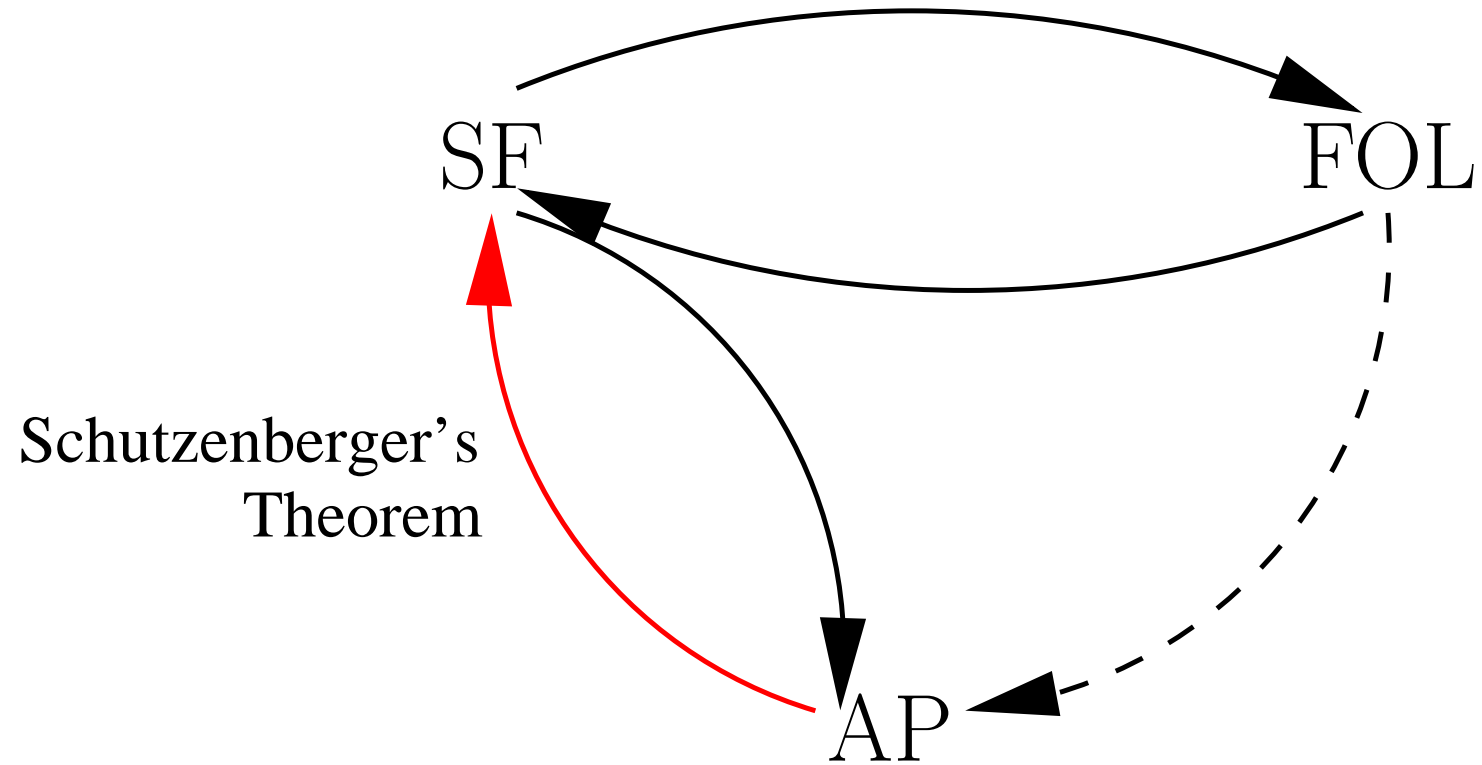
Example 4 $(00)^*1$ *is periodic.*

Given n_0 take the next even number $n \geq n_0$, $u = \epsilon$, $v = 0$ and $t = 1$. Then $uv^n t \in (00)^*1$ and $uv^{n+1} t \notin (00)^*1$. $\square$

Exercise 4 *Is $(00)^*1$ WS1S-definable ?*

Exercise 5 *Is the language $(ab)^*$ periodic or aperiodic ?*

SF · FOL · AP · Schutzenberger's Theorem

From Star-free to Aperiodic

Proposition 6 *If $L \in SF(\Sigma)$ then L is aperiodic.*

Prove the existence of an integer $N(L)$ such that

$$\forall n \geq N(L) \; \forall u \forall v \forall t \; . \; uv^n t \in L \iff uv^{n+1} t \in L$$

. Suppose $v \neq \epsilon$. By induction on the structure of L:

- $\emptyset$: $N(\emptyset) = 0$, since $\forall n \geq 0 \; . \; uv^n t \notin L$

- $\{a\}, a \in \Sigma$: $N(\{a\}) = 2$, since $\forall n \geq 2 \; . \; uv^n t \notin L$

- $\overline{X}$: $N(\overline{X}) = N(X)$, trivial

- $X \cup Y$: $N(X \cup Y) = \max\{N(X), N(Y)\}$, trivial

- XY : $N(XY) = N(X) + N(Y) + 1$, since for all
  $n = n_1 + n_2 + 1 \geq N(X) + N(Y) + 1$, we have either $n_1 \geq N(X)$ or
  $n_2 \geq N(Y)$. Then $uv^n t = (uv^{n_1} r)(sv^{n_2} t)$, where $rs = v$ and
  $uv^{n_1} r \in X$, $sv^{n_2} t \in Y$. If $n_1 \geq N(X)$, $uv^{n_1+1} r \in X \Rightarrow uv^{n+1} t \in XY$ $\square$