

Automata on Finite Words

Definition

A *nondeterministic finite automaton* (NFA) over Σ is a 4-tuple

$A = \langle S, I, T, F \rangle$, where:

- S is a finite set of *states*,
- $I \subseteq S$ is a set of *initial states*,
- $T \subseteq S \times \Sigma \times S$ is a *transition relation*,
- $F \subseteq S$ is a set of *final states*.

We denote $T(s, \alpha) = \{s' \in S \mid (s, \alpha, s') \in T\}$. When T is clear from the context we denote $(s, \alpha, s') \in T$ by $s \xrightarrow{\alpha} s'$.

Runs and Acceptance Conditions

Given a finite word $w \in \Sigma^*$, $w = \alpha_1\alpha_2 \dots \alpha_n$, a *run* of A over w is a finite sequence of states $s_1, s_2, \dots, s_n, s_{n+1}$ such that $s_1 \in I$ and $s_i \xrightarrow{\alpha_i} s_{i+1}$ for all $1 \leq i \leq n$.

A run over w between s_i and s_j is denoted as $s_i \xrightarrow{w} s_j$.

The run is said to be *accepting* if and only if $s_{n+1} \in F$. If A has an accepting run over w , then we say that A *accepts* w .

The language of A , denoted $\mathcal{L}(A)$ is the set of all words accepted by A .

A set of words $S \subseteq \Sigma^*$ is *recognizable* if there exists an automaton A such that $S = \mathcal{L}(A)$.

Determinism and Completeness

Definition 1 An automaton $A = \langle S, I, T, F \rangle$ is **deterministic** (DFA) if and only if $\|I\| \leq 1$ and, for each $s \in S$ and for each $\alpha \in \Sigma$, $\|T(s, \alpha)\| \leq 1$.

If A is deterministic we write $T(s, \alpha) = s'$ instead of $T(s, \alpha) = \{s'\}$.

Definition 2 An automaton $A = \langle S, I, T, F \rangle$ is **complete** if and only if $\|I\| \geq 1$ and, for each $s \in S$ and for each $\alpha \in \Sigma$, $\|T(s, \alpha)\| \geq 1$.

Determinism and Completeness

Proposition 1 *If A is deterministic, then it has **at most one run** for each input word.*

Proposition 2 *If A is complete, then it has **at least one run** for each input word.*

Determinization

Theorem 1 *For every NFA A there exists a DFA A_d such that $\mathcal{L}(A) = \mathcal{L}(A_d)$.*

Let $A_d = \langle 2^S, \{I\}, T_d, \{G \subseteq S \mid G \cap F \neq \emptyset\} \rangle$, where

$$(S_1, \alpha, S_2) \in T_d \iff S_2 = \{s' \mid \exists s \in S_1 . (s, \alpha, s') \in T\}$$

This definition is known as **subset construction**.

Exercise 1 *Let $\Sigma = \{a, b\}$ and $L_n = \{uav \mid u, v \in \Sigma^*, |v| = n - 1\}$, for each integer $n \geq 1$. Build an NFA that recognizes L_n and apply subset construction to it.*

Completion

Lemma 1 *For every NFA A there exists a complete NFA A_c such that $\mathcal{L}(A) = \mathcal{L}(A_c)$.*

Let $A_c = \langle S \cup \{\sigma\}, I, T_c, F \rangle$, where $\sigma \notin S$ is a new **sink state**. The transition relation T_c is defined as:

$$\forall s \in S \forall \alpha \in \Sigma . (s, \alpha, \sigma) \in T_c \iff \forall s' \in S . (s, \alpha, s') \notin T$$

and $\forall \alpha \in \Sigma . (\sigma, \alpha, \sigma) \in T_c$.

Remark: The subset construction yields a complete deterministic automaton, with sink state \emptyset .

Closure Properties

Theorem 2 Let $A_1 = \langle S_1, I_1, T_1, F_1 \rangle$ and $A_2 = \langle S_2, I_2, T_2, F_2 \rangle$ be two NFA, such that $S_1 \cap S_2 = \emptyset$. There exists automata \bar{A}_1 , A_\cup and A_\cap that recognize the languages $\Sigma^* \setminus \mathcal{L}(A_1)$, $\mathcal{L}(A_1) \cup \mathcal{L}(A_2)$, and $\mathcal{L}(A_1) \cap \mathcal{L}(A_2)$, respectively.

Let $A' = \langle S', I', T', F' \rangle$ be the **complete** and **deterministic** (why?) automaton such that $\mathcal{L}(A_1) = \mathcal{L}(A')$, and $\bar{A}_1 = \langle S', I', T', S' \setminus F' \rangle$.

Let $A_\cup = \langle S_1 \cup S_2, I_1 \cup I_2, T_1 \cup T_2, F_1 \cup F_2 \rangle$.

Let $A_\cap = \langle S_1 \times S_2, I_1 \times I_2, T_\cap, F_1 \times F_2 \rangle$ where:

$$(\langle s_1, t_1 \rangle, \alpha, \langle s_2, t_2 \rangle) \in T_\cap \iff (s_1, \alpha, s_2) \in T_1 \text{ and } (t_1, \alpha, t_2) \in T_2$$

On the Exponential Blowup of Complementation

Theorem 3 *For every $n \in \mathbb{N}$, $n \geq 1$, there exists an automaton A , with $\text{size}(A) = n + 1$ such that no deterministic automaton with less than 2^n states recognizes the complement of $\mathcal{L}(A)$.*

Let $\Sigma = \{a, b\}$ and $L_n = \{uav \mid u, v \in \Sigma^*, |v| = n - 1\}$, for all $n \geq 1$.

There exists a NFA with exactly $n + 1$ states which recognizes L_n .

Suppose that $B = \langle S, \{s_0\}, T, F \rangle$, is a (complete) DFA with $\|S\| < 2^n$ that accepts $\Sigma^* \setminus L_n$.

On the Exponential Blowup of Complementation

$\|\{w \in \Sigma^* \mid |w| = n\}\| = 2^n$ and $\|S\| < 2^n$ (by the pigeonhole principle)

$\Rightarrow \exists uav_1, ubv_2 \ . \ |uav_1| = |ubv_2| = n$ and $s \in S \ . \ s_0 \xrightarrow{uav_1} s$ and $s_0 \xrightarrow{ubv_2} s$

Let s_1 be the (unique) state of B such that $s \xrightarrow{u} s_1$.

Since $|uav_1| = n$, then $uav_1u \in L_n \Rightarrow uav_1u \notin \mathcal{L}(B)$, i.e. s is not accepting.

On the other hand, $ubv_2u \notin L_n \Rightarrow ubv_2u \in \mathcal{L}(B)$, i.e. s is accepting,
contradiction.

Projections

Let the input alphabet be $\Sigma = \Sigma_1 \times \Sigma_2$. Any word $w \in \Sigma^*$ can be uniquely identified to a pair $\langle w_1, w_2 \rangle \in \Sigma_1^* \times \Sigma_2^*$ such that $|w_1| = |w_2| = |w|$.

The *projection* operations are

$pr_1(L) = \{u \in \Sigma_1^* \mid \langle u, v \rangle \in L, \text{ for some } v \in \Sigma_2^*\}$ and

$pr_2(L) = \{v \in \Sigma_2^* \mid \langle u, v \rangle \in L, \text{ for some } u \in \Sigma_1^*\}$.

Theorem 4 *If the language $L \subseteq (\Sigma_1 \times \Sigma_2)^*$ is recognizable, then so are the projections $pr_i(L)$, for $i = 1, 2$.*

Remark

The operations of union, intersection and complement correspond to the boolean \vee , \wedge and \neg .

The projection corresponds to the first-order existential quantifier $\exists x$.

The Myhill-Nerode Theorem

Let $A = \langle S, I, T, F \rangle$ be an automaton over the alphabet Σ^* .

Define the relation $\sim_A \subseteq \Sigma^* \times \Sigma^*$ as:

$$u \sim_A v \iff [\forall s, s' \in S . s \xrightarrow{u} s' \iff s \xrightarrow{v} s']$$

\sim_A is an equivalence relation of finite index

Let $L \subseteq \Sigma^*$ be a language. Define the relation $\sim_L \subseteq \Sigma^* \times \Sigma^*$ as:

$$u \sim_L v \iff [\forall w \in \Sigma^* . uw \in L \iff vw \in L]$$

\sim_L is an equivalence relation

The Myhill-Nerode Theorem

Theorem 5 *A language $L \subseteq \Sigma^*$ is recognizable iff \sim_L is of finite index.*

“ \Rightarrow ” Suppose $L = \mathcal{L}(A)$ for some automaton A .

\sim_A is of finite index.

for all $u, v \in \Sigma^*$ we have $u \sim_A v \Rightarrow u \sim_L v$

index of $\sim_L \leq \text{index of } \sim_A < \infty$

The Myhill-Nerode Theorem

“ \Leftarrow ” \sim_L is an equivalence relation of finite index, and let $[u]$ denote the equivalence class of $u \in \Sigma^*$.

$A = \langle S, I, T, F \rangle$, where:

- $S = \{[u] \mid u \in \Sigma^*\},$
- $I = [\epsilon],$
- $[u] \xrightarrow{\alpha} [v] \iff u\alpha \sim_L v,$
- $F = \{[u] \mid u \in L\}.$

For DFA all minimal automata are isomorphic.

For NFA there may be more non-isomorphic minimal automata.

Pumping Lemma

Lemma 2 (Pumping) *Let $A = \langle S, I, T, F \rangle$ be a finite automaton with $\text{size}(A) = n$, and $w \in \mathcal{L}(A)$ be a word of length $|w| \geq n$. Then there exists three words $u, v, t \in \Sigma^*$ such that:*

1. $|v| \geq 1$,
2. $w = uvt$ and,
3. for all $k \geq 0$, $uv^k t \in \mathcal{L}(A)$.

Example

$L = \{a^n b^n \mid n \in \mathbb{N}\}$ is not recognizable:

Suppose that there exists an automaton A with $\text{size}(A) = N$, such that $L = \mathcal{L}(A)$.

Consider the word $a^N b^N \in L = \mathcal{L}(A)$.

There exists words u, v, w such that $|v| \geq 1$, $uvw = a^N b^N$ and $uv^k w \in L$ for all $k \geq 1$.

- $v = a^m$, for some $m \in \mathbb{N}$.
- $v = a^m b^p$ for some $m, p \in \mathbb{N}$.
- $v = b^m$, for some $m \in \mathbb{N}$.

Decidability

Given nondeterministic finite automata A and B :

- **Emptiness** $\mathcal{L}(A) = \emptyset$?
- **Inclusion** $\mathcal{L}(A) \subseteq \mathcal{L}(B)$?
- **Equivalence** $\mathcal{L}(A) = \mathcal{L}(B)$?
- **Infinity** $\|\mathcal{L}(A)\| < \infty$?
- **Universality** $\mathcal{L}(A) = \Sigma^*$?

Emptiness

Theorem 6 *Let A be an automaton with $\text{size}(A) = n$. If $\mathcal{L}(A) \neq \emptyset$, then there exists a word of length less than n that is accepted by A .*

Let u be the shortest word in $\mathcal{L}(A)$.

If $|u| < n$ we are done.

If $|u| \geq n$, there exists $u_1, v, u_2 \in \Sigma^*$ such that $|v| > 1$ and $u_1vu_2 = u$.

Then $u_1u_2 \in \mathcal{L}(A)$ and $|u_1u_2| < |u_1vu_2|$, contradiction.

Everything is decidable

Theorem 7 *The emptiness, equality, infinity and universality problems are decidable for automata on finite words.*

Although complexity varies from problem to problem:

- **Emptiness** ($\mathcal{L}(A) = \emptyset$) belongs to NLOGSPACE
- **Inclusion** ($\mathcal{L}(A) \subseteq \mathcal{L}(B)$) is PSPACE-complete
- **Equivalence** ($\mathcal{L}(A) = \mathcal{L}(B)$) is PSPACE-complete
- **Infinity** ($\|\mathcal{L}(A)\| < \infty$) belongs to NLOGSPACE
- **Universality** ($\mathcal{L}(A) = \Sigma^*$) is PSPACE-complete

Automata on Finite Words and WS1S

WS1S

Let $\Sigma = \{a, b, \dots\}$ be a finite alphabet.

Any finite word $w \in \Sigma^*$ induces the *finite* sets $p_a = \{p \mid w(p) = a\}$.

- $x \leq y$: x is less than y ,
- $s(x) = y$: y is the successor of x ,
- $p_a(x)$: a occurs at position x in w

Remember that \leq and $s(\cdot)$ can be defined one from another.

Problem Statement

Given a sentence φ in WS1S, let $\mathcal{L}(\varphi) = \{w \mid \mathfrak{m}_w \models \varphi\}$, where $\mathfrak{m}_w = \langle \text{dom}(w), \{\bar{p}_a\}_{a \in \Sigma}, \leq \rangle$, such that:

- $\text{dom}(w) = \{0, 1, \dots, n-1\}$,
- $\bar{p}_a = \{x \in \text{dom}(w) \mid w(x) = a\}$,

A language $L \subseteq \Sigma^*$ is said to be *WS1S-definable* iff there exists a WS1S sentence φ such that $L = \mathcal{L}(\varphi)$.

1. Given A build φ_A such that $\mathcal{L}(A) = \mathcal{L}(\varphi)$
2. Given φ build A_φ such that $\mathcal{L}(A) = \mathcal{L}(\varphi)$

The recognizable and WS1S-definable languages coincide

Coding of Σ

Let $m \in \mathbb{N}$ be the smallest number such that $\|\Sigma\| \leq 2^m$.

W.l.o.g. assume that $\Sigma = \{0, 1\}^m$, and let $X_1 \dots X_p, x_{p+1}, \dots, x_m$

A word $w \in \Sigma^*$ induces an *interpretation* of $X_1 \dots X_p, x_{p+1}, \dots, x_m$:

- $i \in \iota_w(X_j)$ iff the j -th element of w_i is 1, and
- $\iota_w(x_j) = i$ iff w_i has 1 on the j -th position and, for all $k \neq i$ w_k has 0 on the j -th position.

Example

Example 1 Let $\Sigma = \{a, b, c, d\}$, encoded as $a = (00)$, $b = (01)$, $c = (10)$ and $d = (11)$. Then the word $abbaacdd$ induces the valuation $X_1 = \{5, 6, 7\}$, $X_2 = \{1, 2, 6, 7\}$. \square

From Automata to Formulae

Let $A = \langle S, I, T, F \rangle$ with $S = \{s_1, \dots, s_p\}$, and $\Sigma = \{0, 1\}^m$.

Build $\Phi_A(X_1, \dots, X_m)$ such that $\forall w \in \Sigma^* . w \in \mathcal{L}(A) \iff \llbracket \Phi_A \rrbracket_{\iota_w}^{\mathbf{m}_w} = \text{true}$

Let $a \in \{0, 1\}^m$. Let $\Phi_a(x, X_1, \dots, X_m)$ be the conjunction of:

- $X_i(x)$ if the $a_i = 1$, and
- $\neg X_i(x)$ otherwise.

For all $w \in \Sigma^*$ we have $\mathbf{m}_w \models \forall x . \bigvee_{a \in \Sigma} \Phi_a(x, \vec{X})$

Notice that $\Phi_a \wedge \Phi_b$ is unsatisfiable, for $a \neq b$.

Coding of S

Let $\{Y_0, \dots, Y_p\}$ be set variables.

Y_i is the set of all positions labeled by A with state s_i during some run

$$\Phi_S(Y_1, \dots, Y_p) \quad : \quad \forall z \ . \quad \bigvee_{1 \leq i \leq p} Y_i(z) \ \wedge \quad \bigwedge_{1 \leq i < j \leq p} \neg \exists z \ . \ Y_i(z) \wedge Y_j(z)$$

Coding of I

Every run starts from an initial state:

$$\Phi_I(Y_1, \dots, Y_p) \quad : \quad \exists x \forall y . x \leq y \wedge \bigvee_{s_i \in I} Y_i(x)$$

Coding of T

Consider the transition $s_i \xrightarrow{a} s_j$:

$$\Phi_T(X_1, \dots, X_m, Y_1, \dots, Y_p) : \forall x . \neg last(x) \wedge Y_i(x) \wedge \Phi_a(x, \vec{X}) \rightarrow \bigvee_{(s_i, a, s_j) \in T} Y_j(s(x))$$

where $last(x) = \forall y . y \leq x$

Coding of F

The last state on the run is a final state:

$$\Phi_F(Y_1, \dots, Y_p) \quad : \quad \exists x . \textit{last}(x) \wedge \bigvee_{s_i \in F} Y_i(x)$$

$$\Phi_A = \exists Y_1 \dots \exists Y_p . \Phi_S \wedge \Phi_I \wedge \Phi_T \wedge \Phi_F$$

From Formulae to Automata

Let $\Phi(X_1, \dots, X_p, x_{p+1}, \dots, x_m)$ be a WS1S formula.

Build an automaton A_Φ such that $\forall w \in \Sigma^* . w \in \mathcal{L}(A) \iff \llbracket \Phi \rrbracket_{\iota_w}^{\mathfrak{m}_w} = \text{true}$

Let $\Phi(X_1, X_2, x_3, x_4)$ be:

1. $X_1(x_3)$
2. $x_3 \leq x_4$
3. $X_1 = X_2$

From Formulae to Automata

A_Φ is built by induction on the structure of Φ :

- for $\Phi = \phi_1 \wedge \phi_2$ we have $\mathcal{L}(A_\Phi) = \mathcal{L}(A_{\phi_1}) \cap \mathcal{L}(A_{\phi_2})$
- for $\Phi = \phi_1 \vee \phi_2$ we have $\mathcal{L}(A_\Phi) = \mathcal{L}(A_{\phi_1}) \cup \mathcal{L}(A_{\phi_2})$
- for $\Phi = \neg\phi$ we have $\mathcal{L}(A_\Phi) = \overline{\mathcal{L}(A_\phi)}$
- for $\Phi = \exists X_i . \phi$, we have $\mathcal{L}(A_\Phi) = pr_i(\mathcal{L}(A_\phi))$.

Consequences

Theorem 8 *A language $L \subseteq \Sigma^*$ is definable in WS1S iff it is recognizable.*

Corollary 1 *The SAT problem for WS1S is decidable.*

Exercise 2 *Prove that there is no WS1S formula $\varphi(x, y, z)$ that defines the relation $\{(m, n, p) \in \mathbb{N}^3 \mid m + n = p\}$.*

Regular, Star Free and Aperiodic Languages

Regular Languages

Let Σ be an alphabet, and $X, Y \subseteq \Sigma^*$

$$XY = \{xy \mid x \in X \text{ and } y \in Y\}$$

$$X^* = \{x_1 \dots x_n \mid n \geq 0, x_1, \dots, x_n \in X\}$$

The class of *regular languages* $\mathcal{R}(\Sigma)$ is the smallest class of languages $L \subseteq \Sigma^*$ such that:

- $\emptyset, \{\epsilon\} \in \mathcal{R}(\Sigma)$
- $\{\alpha\} \in \mathcal{R}(\Sigma)$, for all $\alpha \in \Sigma$
- if $X, Y \in \mathcal{R}(\Sigma)$ then $X \cup Y, XY, X^* \in \mathcal{R}(\Sigma)$

Regular, rational and recognizable languages

Theorem 9 (Kleene) *A set of finite words is recognizable if and only if it is regular.*

Proof in every textbook.

Rational = regular, in older books e.g.

Samuel Eilenberg. *Automata, Languages and Machines*. Academic Press, 1974

Star Free Languages

The class of *star-free languages* is the smallest class $SF(\Sigma)$ of languages $L \in \Sigma^*$ such that:

- $\emptyset, \{\epsilon\} \in SF(\Sigma)$ and $\{a\} \in SF(\Sigma)$ for all $a \in \Sigma$
- if $X, Y \in SF(\Sigma)$ then $X \cup Y, XY, \overline{X} \in SF(\Sigma)$

Example 2

- $\Sigma^* = \overline{\emptyset}$ is star-free
- if $B \subset \Sigma$, then $\Sigma^* B \Sigma^* = \bigcup_{b \in B} \Sigma^* b \Sigma^*$ is star-free
- if $B \subset \Sigma$, then $B^* = \overline{\Sigma^* \overline{B} \Sigma^*}$ is star-free
- if $\Sigma = \{a, b\}$, then $(ab)^* = \overline{b \Sigma^* \cup \Sigma^* a \cup \Sigma^* a a \Sigma^* \cup \Sigma^* b b \Sigma^*}$ is star-free

Exercise 3 If $\Sigma = \{a, b, c\}$, write $(ab)^*$ as a star-free language.

The Splitting Lemma

Lemma 3 *Let $A, B \subseteq \Sigma$ be subalphabets such that $A \cap B = \emptyset$. Then, for each star-free language $L \in SF(\Sigma)$, we have:*

$$L \cap B^*AB^* = \bigcup_{1 \leq i \leq n} K_i a_i L_i$$

where $a_i \in A$ and $K_i, L_i \in SF(B)$, for all $1 \leq i \leq n$.

W.l.o.g. we prove the case $A = \{a\}$ (why?) by induction on L :

- If $L = \{a\}$ then $L \cap B^*AB^* = \{\epsilon\}a\{\epsilon\}$.
- If $L = \{a'\}$, $a' \neq a$, then $L \cap B^*AB^* = \emptyset a \emptyset$.
- If $L = \Sigma^*$ then $L \cap B^*AB^* = B^*AB^*$.
- If $L = L_1 \cup L_2$ then $L \cap B^*AB^* = (L_1 \cap B^*AB^*) \cup (L_2 \cap B^*AB^*)$.

The Splitting Lemma

$$L \cap B^*AB^* = \bigcup_{1 \leq i \leq n} K_i a_i L_i$$

- If $L = L_1 \cdot L_2$ then

$$L \cap B^*AB^* = (L_1 \cap B^*) \cdot (L_2 \cap B^*AB^*) \cup (L_1 \cap B^*AB^*) \cdot (L_2 \cap B).$$

- Else, if $L = \Sigma^* \setminus L'$, by the inductive hypothesis $L' = \bigcup_{1 \leq i \leq n} K'_i a L'_i$.

We assume w.l.o.g that $\{K'_i\}_{i=1}^n$ form a partition of B^* :

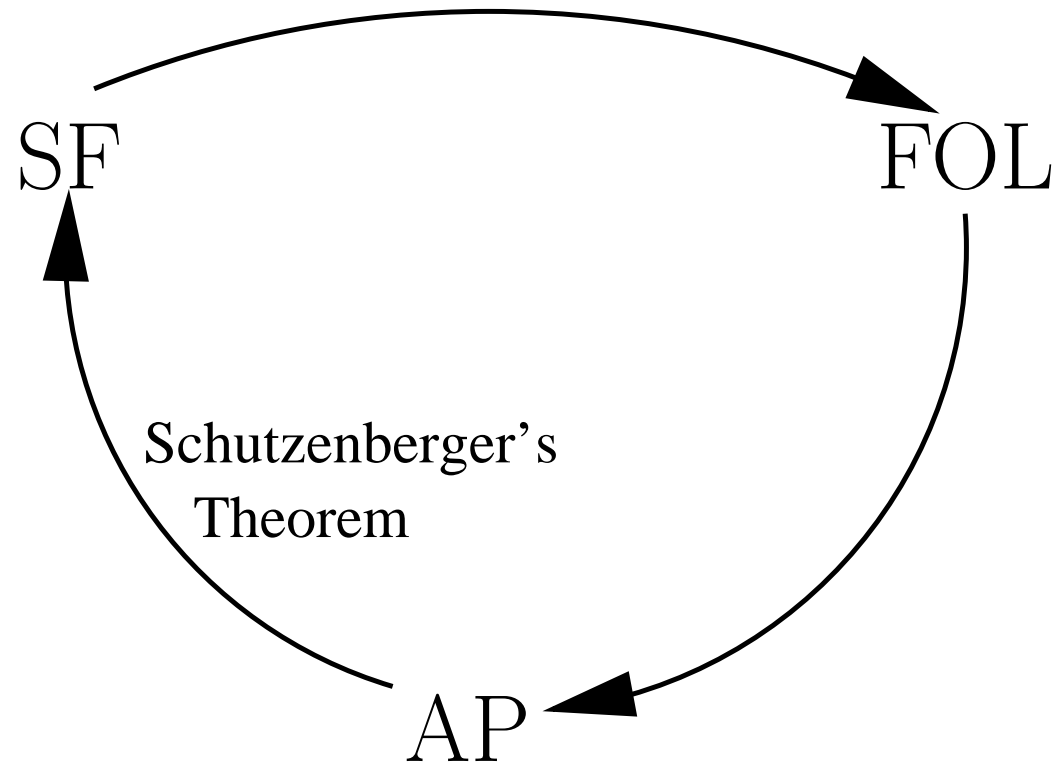
- if $K'_i \cap K'_j \neq \emptyset$, rewrite

$$K'_i a L'_i \cup K'_j a L'_j = (K'_i \setminus K'_j) a L'_i \cup (K'_j \setminus K'_i) a L'_j \cup (K'_i \cap K'_j) a (L'_i \cup L'_j)$$

- if $\bigcup_{i=1}^n K'_i \subsetneq B^*$, add $(B^* \setminus \bigcup_{i=1}^n K'_i) a \emptyset$ to $\{K'_i\}_{i=1}^n$

$$(\Sigma^* \setminus L') \cap B^* a B^* = \bigcup_{i=1}^n K'_i a (B^* \setminus L'_i)$$

SF = FOL (= AP)



Subword Formulae

Let $w = a_0a_1 \dots a_{n-1}$ be a finite word, and $w(i, j) = a_ia_{i+1} \dots a_{j-1}$ be a subword of w , $0 \leq i < n$ and $0 \leq j \leq n$, $i < j$.

Proposition 3 *For each FOL sentence φ there exists a formula $\varphi[x, y]$ such that, for each $w \in \Sigma^*$ and each $0 \leq i < j \leq |w|$:*

$$\mathfrak{m}_{w(i,j)} \models \varphi \iff \llbracket \varphi[x, y] \rrbracket_{[x \leftarrow i][y \leftarrow j]}^{\mathfrak{m}_w} = \mathbf{true}$$

By induction on the structure of φ :

$$\begin{aligned} (\neg\varphi)[x, y] &= \neg(\varphi[x, y]) \\ (\varphi \wedge \psi)[x, y] &= (\varphi[x, y]) \wedge (\psi[x, y]) \\ (\exists z.\varphi)[x, y] &= \exists z . x \leq z \wedge z < y \wedge \varphi[x, y] \end{aligned}$$

Star Free Languages are FOL-definable

For each $L \in SF(\Sigma)$, there exists an FOL **sentence** φ_L such that:

$$L = \{w \in \Sigma^* \mid \mathfrak{m}_w \models \varphi_L\}$$

By induction on the structure of L :

$$\emptyset = \{w \in \Sigma^* \mid \mathfrak{m}_w \models \perp\} \qquad \{a\} = \{w \in \Sigma^* \mid \mathfrak{m}_u \models p_a(0) \wedge len(1)\}$$

$$X \cup Y = \{w \in \Sigma^* \mid \mathfrak{m}_u \models \varphi_X \vee \varphi_Y\} \qquad \overline{X} = \{w \in \Sigma^* \mid \mathfrak{m}_u \models \neg \varphi_X\}$$

$$X \cdot Y = \{w \in \Sigma^* \mid \mathfrak{m}_u \models \exists y \exists z . 0 \leq y < z \wedge \varphi_X[0, y] \wedge \varphi_Y[y, z] \wedge len(z)\}$$

FOL-definable Languages are Star Free

Let φ be an FOL formula with $FV(\varphi) = V$ and let $\Sigma_V = \Sigma \times \{0, 1\}^V$.

Encode each pair (w, ι) , with $\iota : V \rightarrow [0, |w| - 1]$ as a word $\overline{(w, \iota)} \in \Sigma_V^*$:

$$\overline{(a_0 \dots a_{k-1}, \iota)} = (a_0, \tau_0) \dots (a_{k-1}, \tau_{k-1}), \quad \tau_i(x) = 1 \iff \iota(x) = i$$

and let $\mathcal{N}_V = \{\overline{(w, \iota)} \mid w \in \Sigma^*, \iota : V \rightarrow [0, |w| - 1]\}$.

Let $\Sigma_V^{x=i} = \{(a, \tau) \mid \tau(x) = i\}$, for $i = 0, 1$

$$\mathcal{N}_V = \bigcap_{x \in V} (\Sigma_V^{x=0})^* (\Sigma_V^{x=1}) (\Sigma_V^{x=0})^* \in SF(\Sigma_V)$$

FOL-definable Languages are Star Free

$$\begin{aligned}\llbracket p_a(x) \rrbracket_V &= \{ \overline{(w, \iota)} \in \mathcal{N}_V \mid w = a_0 \dots a_{k-1}, a_{\iota(x)} = a \} \\ \llbracket x \leq y \rrbracket_V &= \{ \overline{(w, \iota)} \in \mathcal{N}_V \mid \iota(x) \leq \iota(y) \} \\ \llbracket \phi \vee \psi \rrbracket_V &= \llbracket \phi \rrbracket_V \cup \llbracket \psi \rrbracket_V \\ \llbracket \neg \phi \rrbracket_V &= \mathcal{N}_V \setminus \llbracket \phi \rrbracket_V \\ \llbracket \exists x . \phi \rrbracket_V &= \{ \overline{(w, \iota)} \in \mathcal{N}_V \mid \exists i \in [0, |w| - 1] . \overline{(w, \iota[x \leftarrow i])} \in \llbracket \phi \rrbracket_{V \cup \{x\}} \end{aligned}$$

Proposition 4 *If $\varphi \in FOL$ and $FV(\varphi) \subseteq V$, then $\llbracket \varphi \rrbracket_V \in SF(\Sigma_V)$.*

$$\begin{aligned}\llbracket p_a(x) \rrbracket_V &= \mathcal{N}_V \cap (\Sigma_V^* \cdot \{(a, \tau) \mid \tau(x) = 1\} \cdot \Sigma_V^*) \\ \llbracket x \leq y \rrbracket_V &= \mathcal{N}_V \cap (\Sigma_V^* \cdot \Sigma_V^{x=1} \cdot \Sigma_V^* \cdot \Sigma_V^{y=1} \cdot \Sigma_V^*)\end{aligned}$$

FOL-definable Languages are Star Free

Proposition 5 *If $\varphi \in FOL$ and $FV(\varphi) \subseteq V$, then $\llbracket \varphi \rrbracket_V \in SF(\Sigma_V)$.*

If $\varphi = \exists x . \phi$, we assume w.l.o.g. that $x \notin V$ (α -conversion)

$$\begin{aligned}\llbracket \phi \rrbracket_{V \cup \{x\}} &= \llbracket \phi \rrbracket_{V \cup \{x\}} \cap (\Sigma_{V \cup \{x\}}^{x=0})^* (\Sigma_{V \cup \{x\}}^{x=1}) (\Sigma_{V \cup \{x\}}^{x=0})^* \\ &= \bigcup_{i=1}^n K'_i a'_i L'_i \text{ (Splitting Lemma)}\end{aligned}$$

where $K'_i, L'_i \in SF(\Sigma_{V \cup \{x\}}^{x=0})$ and $a'_i \in \Sigma_{V \cup \{x\}}^{x=1}$, for all $1 \leq i \leq n$

Let $\pi : \Sigma_{V \cup \{x\}}^{x=0} \rightarrow \Sigma_V$ be the **bijective** (why?) renaming $(a, \tau) \mapsto^\pi (a, \tau \downarrow_V)$

Let $K_i = \pi(K'_i)$, $L_i = \pi(L'_i)$ and $a_i = (a, \tau \downarrow_V) \iff a_i = (a, \tau)$

$$\llbracket \exists x . \phi \rrbracket_V = \bigcup_{i=1}^n K_i a_i L_i$$

NB: SF languages are preserved by bijective renamings (why bijective ?)

Aperiodic Languages

Definition 3 A language $L \subseteq \Sigma^*$ is said to be **aperiodic** iff:

$$\exists n_0 \forall n \geq n_0 \forall u, v, t \in \Sigma^* . uv^n t \in L \iff uv^{n+1} t \in L$$

n_0 is called the **index** of L .

Example 3 0^*1^* is aperiodic. Let $n_0 = 2$. We have three cases:

1. $u, v \in 0^*$ and $t \in 0^*1^*$:

$$\forall n \geq 2 . uv^n t \in L$$

2. $u \in 0^*, v \in 0^+1^+$ and $t \in 1^*$:

$$\forall n \geq 2 . uv^n t \notin L$$

3. $u \in 0^*1^*, v \in 1^*$ and $t \in 1^*$:

$$\forall n \geq 2 . uv^n t \in L$$

Periodic Languages

Conversely, a language $L \subseteq \Sigma^*$ is said to be *periodic* iff:

$$\forall n_0 \exists n \geq n_0 \exists u, v, t \in \Sigma^* . (uv^n t \notin L \wedge uv^{n+1} t \in L) \vee (uv^n t \in L \wedge uv^{n+1} t \notin L)$$

Example 4 $(00)^*1$ is periodic.

Given n_0 take the next even number $n \geq n_0$, $u = \epsilon$, $v = 0$ and $t = 1$. Then $uv^n t \in (00)^*1$ and $uv^{n+1} t \notin (00)^*1$. \square

Exercise 4 Is $(00)^*1$ WS1S-definable ?

Exercise 5 Is the language $(ab)^*$ periodic or aperiodic ?

From Star-free to Aperiodic

Proposition 6 *If $L \in SF(\Sigma)$ then L is aperiodic.*

Prove the existence of an integer $N(L)$ such that

$$\forall n \geq N(L) \forall u \forall v \forall t . uv^n t \in L \iff uv^{n+1} t \in L$$

. Suppose $v \neq \epsilon$. By induction on the structure of L :

- $\emptyset : N(\emptyset) = 0$, since $\forall n \geq 0 . uv^n t \notin L$
- $\{a\}, a \in \Sigma : N(\{a\}) = 2$, since $\forall n \geq 2 . uv^n t \notin L$
- $\overline{X} : N(\overline{X}) = N(X)$, trivial
- $X \cup Y : N(X \cup Y) = \max\{N(X), N(Y)\}$, trivial
- $XY : N(XY) = N(X) + N(Y) + 1$, since for all $n = n_1 + n_2 + 1 \geq N(X) + N(Y) + 1$, we have either $n_1 \geq N(X)$ or $n_2 \geq N(Y)$. Then $uv^n t = (uv^{n_1} r)(sv^{n_2} t)$, where $rs = v$ and $uv^{n_1} r \in X, sv^{n_2} t \in Y$. If $n_1 \geq N(X)$, $uv^{n_1+1} r \in X \Rightarrow uv^{n+1} t \in XY \quad \square$

The Big Picture

