Automata on Finite Words

Definition

A *nondeterministic finite automaton* (NFA) over Σ is a 4-tuple $A = \langle S, I, T, F \rangle$, where:

- S is a finite set of *states*,
- $I \subseteq S$ is a set of *initial states*,
- $T \subseteq S \times \Sigma \times S$ is a *transition relation*,
- $F \subseteq S$ is a set of *final states*.

We denote $T(s, \alpha) = \{s' \in S \mid (s, \alpha, s') \in T\}$. When T is clear from the context we denote $(s, \alpha, s') \in T$ by $s \xrightarrow{\alpha} s'$.

Determinism and Completeness

Definition 1 *An automaton* $A = \langle S, I, T, F \rangle$ *is* deterministic *(DFA) iff* $\|I\| = 1$ *and, for each* $s \in S$ *and for each* $\alpha \in \Sigma$, $\|T(s, \alpha)\| \leq 1$.

If A is deterministic we write $T(s, \alpha) = s'$ instead of $T(s, \alpha) = \{s'\}$.

Definition 2 *An automaton* $A = \langle S, I, T, F \rangle$ *is* complete *iff* $\|I\| \geq 1$ *and, for each* $s \in S$ *and for each* $\alpha \in \Sigma$, $\|T(s, \alpha)\| \geq 1$.

Runs and Acceptance Conditions

Given a finite word $w \in \Sigma^*$, $w = \alpha_1\alpha_2 \ldots \alpha_n$, a *run* of A over w is a finite sequence of states $s_1, s_2, \ldots, s_n, s_{n+1}$ such that $s_1 \in I$ and $s_i \xrightarrow{\alpha_i} s_{i+1}$ for all $1 \leq i \leq n$.

A run over w between s_i and s_j is denoted as $s_i \xrightarrow{w} s_j$.

The run is said to be *accepting* iff $s_{n+1} \in F$. If A has an accepting run over w, then we say that A *accepts* w.

The language of A, denoted $\mathcal{L}(A)$ is the set of all words accepted by A.

A set of words $S \subseteq \Sigma^*$ is *recognizable* if there exists an automaton A such that $S = \mathcal{L}(A)$.

Determinism, Completeness, again

Proposition 1 *If A is deterministic, then it has at most one run for each input word.*

Proposition 2 *If A is complete, then it has at least one run for each input word.*

Determinization

Theorem 1 *For every NFA A there exists a DFA A_d such that $\mathcal{L}(A) = \mathcal{L}(A_d)$.*

Let $A_d = \langle 2^S, \{I\}, T_d, \{G \subseteq S \mid G \cap F \neq \emptyset\}\rangle$, where

$$(S_1, \alpha, S_2) \in T_d \iff S_2 = \{s' \mid \exists s \in S_1 \,.\, (s, \alpha, s') \in T\}$$

This definition is known as subset construction.

Exercise 1 *Let $\Sigma = \{a, b\}$ and $L_n = \{uav \mid u, v \in \Sigma^*,\ |v| = n - 1\}$, for each integer $n \geq 1$. Build an NFA that recognizes L_n and apply subset construction to it.*

Completion

Lemma 1 *For every NFA A there exists a complete NFA A_c such that $\mathcal{L}(A) = \mathcal{L}(A_c)$.*

Let $A_c = \langle S \cup \{\sigma\}, I, T_c, F\rangle$, where $\sigma \notin S$ is a new sink state. The transition relation T_c is defined as:

$$\forall s \in S \forall \alpha \in \Sigma \,.\, (s, \alpha, \sigma) \in T_c \iff \forall s' \in S \,.\, (s, \alpha, s') \notin T$$

and $\forall \alpha \in \Sigma \,.\, (\sigma, \alpha, \sigma) \in T_c$.

Remark: The subset construction yields a complete deterministic automaton, with sink state $\emptyset$.

Closure Properties

Theorem 2 *Let $A_1 = \langle S_1, I_1, T_1, F_1 \rangle$ and $A_2 = \langle S_2, I_2, T_2, F_2 \rangle$ be two NFA, such that $S_1 \cap S_2 = \emptyset$. There exists automata $\bar{A}_1$, $A_\cup$ and $A_\cap$ that recognize the languages $\Sigma^* \setminus \mathcal{L}(A_1)$, $\mathcal{L}(A_1) \cup \mathcal{L}(A_2)$, and $\mathcal{L}(A_1) \cap \mathcal{L}(A_2)$, respectivelly.*

Let $A' = \langle S', I', T', F' \rangle$ be the complete and deterministic (why?) automaton such that $\mathcal{L}(A_1) = \mathcal{L}(A')$, and $\bar{A}_1 = \langle S', I', T', S' \setminus F' \rangle$.

Let $A_\cup = \langle S_1 \cup S_2, I_1 \cup I_2, T_1 \cup T_2, F_1 \cup F_2 \rangle$.

Let $A_\cap = \langle S_1 \times S_2, I_1 \times I_2, T_\cap, F_1 \times F_2 \rangle$ where:

$$(\langle s_1, t_1 \rangle, \alpha, \langle s_2, t_2 \rangle) \in T_\cap \iff (s_1, \alpha, s_2) \in T_1 \text{ and } (t_1, \alpha, t_2) \in T_2$$

On the Exponential Blowup of Complementation

Theorem 3 *For every $n \in \mathbb{N}$, $n \geq 1$, there exists an automaton A, with $size(A) = n + 1$ such that no deterministic automaton with less than 2^n states recognizes the complement of $\mathcal{L}(A)$.*

Let $\Sigma = \{a, b\}$ and $L_n = \{uav \mid u, v \in \Sigma^*, |v| = n - 1\}$, for all $n \geq 1$.

There exists a NFA with exactly $n + 1$ states which recognizes L_n.

Suppose that $B = \langle S, \{s_0\}, T, F \rangle$, is a (complete) DFA with $\|S\| < 2^n$ that accepts $\Sigma^* \setminus L_n$.

On the Exponential Blowup of Complementation

$\|\{w \in \Sigma^* \mid |w| = n\}\| = 2^n$ and $\|S\| < 2^n$ (by the pigeonhole principle)

$\Rightarrow \exists uav_1, ubv_2 \, . \, |uav_1| = |ubv_2| = n$ and $s \in S \, . \, s_0 \xrightarrow{uav_1} s$ and $s_0 \xrightarrow{ubv_2} s$

Let s_1 be the (unique) state of B such that $s \xrightarrow{u} s_1$.

Since $|uav_1| = n$, then $uav_1u \in L_n \Rightarrow uav_1u \notin \mathcal{L}(B)$, i.e. s is not accepting.

On the other hand, $ubv_2u \notin L_n \Rightarrow ubv_2u \in \mathcal{L}(B)$, i.e. s is accepting, contradiction.

Projections

Let the input alphabet $\Sigma = \Sigma_1 \times \Sigma_2$. Any word $w \in \Sigma^*$ can be uniquely identified to a pair $\langle w_1, w_2 \rangle \in \Sigma_1^* \times \Sigma_2^*$ such that $|w_1| = |w_2| = |w|$.

The *projection* operations are
$pr_1(L) = \{u \in \Sigma_1^* \mid \langle u, v \rangle \in L,\ \text{for some } v \in \Sigma_2^*\}$ and
$pr_2(L) = \{v \in \Sigma_2^* \mid \langle u, v \rangle \in L,\ \text{for some } u \in \Sigma_1^*\}$.

Theorem 4 *If the language $L \subseteq (\Sigma_1 \times \Sigma_2)^*$ is recognizable, then so are the projections $pr_i(L)$, for $i = 1, 2$.*

Remark

The operations of union, intersection and complement correspond to the boolean $\vee$, $\wedge$ and $\neg$.

The projection corresponds to the first-order existential quantifier $\exists x$.

The Myhill-Nerode Theorem

Let $A = \langle S, I, T, F \rangle$ be an automaton over the alphabet Σ^*.

Define the relation $\sim_A \subseteq \Sigma^* \times \Sigma^*$ as:

$$u \sim_A v \iff [\forall s, s' \in S \;.\; s \xrightarrow{u} s' \iff s \xrightarrow{v} s']$$

$\sim_A$ is an equivalence relation of finite index

Let $L \subseteq \Sigma^*$ be a language. Define the relation $\sim_L \subseteq \Sigma^* \times \Sigma^*$ as:

$$u \sim_L v \iff [\forall w \in \Sigma^* \;.\; uw \in L \iff vw \in L]$$

$\sim_L$ is an equivalence relation

The Myhill-Nerode Theorem

Theorem 5 *A language $L \subseteq \Sigma^*$ is recognizable iff $\sim_L$ is of finite index.*

"$\Rightarrow$" Suppose $L = \mathcal{L}(A)$ for some automaton A.

$\sim_A$ is of finite index.

for all $u, v \in \Sigma^*$ we have $u \sim_A v \Rightarrow u \sim_L v$

index of $\sim_L \leq$ index of $\sim_A < \infty$

The Myhill-Nerode Theorem

"$\Leftarrow$" $\sim_L$ is an equivalence relation of finite index, and let $[u]$ denote the equivalence class of $u \in \Sigma^*$.

$A = \langle S, I, T, F \rangle$, where:

- $S = \{[u] \mid u \in \Sigma^*\}$,
- $I = [\epsilon]$,
- $[u] \xrightarrow{\alpha} [v] \iff u\alpha \sim_L v$,
- $F = \{[u] \mid u \in L\}$.

Isomorphism and Canonical Automata

Two automata $A_i = \langle S_i, I_i, T_i, F_i \rangle$, $i = 1, 2$ are said to be *isomorphic* iff there exists a bijection $h : S_1 \to S_2$ such that, for all $s, s' \in S_1$ and for all $\alpha \in \Sigma$ we have :

- $s \in I_1 \iff h(s) \in I_2$,
- $(s, \alpha, s') \in T_1 \iff (h(s), \alpha, h(s')) \in T_2$,
- $s \in F_1 \iff h(s) \in F_2$.

For DFA all minimal automata are isomorphic.

For NFA there may be more non-isomorphic minimal automata.

Pumping Lemma

Lemma 2 (Pumping) *Let $A = \langle S, I, T, F \rangle$ be a finite automaton with $size(A) = n$, and $w \in \mathcal{L}(A)$ be a word of length $|w| \geq n$. Then there exists three words $u, v, t \in \Sigma^*$ such that:*

1. $|v| \geq 1$,
2. $w = uvt$ *and,*
3. *for all* $k \geq 0$, $uv^k t \in \mathcal{L}(A)$.

Example

$L = \{a^n b^n \mid n \in \mathbb{N}\}$ is not recognizable:

Suppose that there exists an automaton A with $size(A) = N$, such that $L = \mathcal{L}(A)$.

Consider the word $a^N b^N \in L = \mathcal{L}(A)$.

There exists words u, v, w such that $|v| \geq 1$, $uvw = a^N b^N$ and $uv^k w \in L$ for all $k \geq 1$.

- $v = a^m$, for some $m \in \mathbb{N}$.
- $v = a^m b^p$ for some $m, p \in \mathbb{N}$.
- $v = b^m$, for some $m \in \mathbb{N}$.

Decidability

Given nondeterministic finite automata A and B:

- **Emptiness** $\mathcal{L}(A) = \emptyset$?
- **Inclusion** $\mathcal{L}(A) \subseteq \mathcal{L}(B)$?
- **Equivalence** $\mathcal{L}(A) = \mathcal{L}(B)$?
- **Infinity** $\|\mathcal{L}(A)\| < \infty$?
- **Universality** $\mathcal{L}(A) = \Sigma^*$?

Emptiness

Theorem 6 *Let A be an automaton with size$(A) = n$. If $\mathcal{L}(A) \neq \emptyset$, then there exists a word of length less than n that is accepted by A.*

Let u be the shortest word in $\mathcal{L}(A)$.

If $|u| < n$ we are done.

If $|u| \geq n$, there exists $u_1, v, u_2 \in \Sigma^*$ such that $|v| > 1$ and $u_1 v u_2 = u$.

Then $u_1 u_2 \in \mathcal{L}(A)$ and $|u_1 u_2| < |u_1 v u_2|$, contradiction.

Everything is decidable

Theorem 7 *The emptiness, equality, infinity and universality problems are decidable for automata on finite words.*

Although complexity varies from problem to problem:

- **Emptiness** ($\mathcal{L}(A) = \emptyset$) belongs to NLOGSPACE
- **Inclusion** ($\mathcal{L}(A) \subseteq \mathcal{L}(B)$) is PSPACE-complete
- **Equivalence** ($\mathcal{L}(A) = \mathcal{L}(B)$) is PSPACE-complete
- **Infinity** ($\|\mathcal{L}(A)\| < \infty$) belongs to NLOGSPACE
- **Universality** ($\mathcal{L}(A) = \Sigma^*$) is PSPACE-complete

Automata on Finite Words and WS1S

WS1S

Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet.

Any finite word $w \in \Sigma^*$ induces the *finite* sets $p_a = \{p \mid w(p) = a\}$.

- $x \leq y$: x is less than y,
- $s(x) = y$: y is the successor of x,
- $p_a(x)$: a occurs at position x in w

Remember that $\leq$ and $s(.)$ can be defined one from another.

Problem Statement

Given a sentence φ in WS1S, let $\mathcal{L}(\varphi) = \{w \mid \mathfrak{m}_w \models \varphi\}$, where $\mathfrak{m}_w = \langle dom(w),\ \{\bar{p_a}\}_{a \in \Sigma},\ \leq \rangle$, such that:

- $dom(w) = \{0, 1, \ldots, n-1\}$,
- $\bar{p_a} = \{x \in dom(w) \mid w(x) = a\}$,

A language $L \subseteq \Sigma^*$ is said to be WS1S-*definable* iff there exists a WS1S sentence φ such that $L = \mathcal{L}(\varphi)$.

1. Given A build φ_A such that $\mathcal{L}(A) = \mathcal{L}(\varphi)$
2. Given φ build A_φ such that $\mathcal{L}(A) = \mathcal{L}(\varphi)$

The recognizable and WS1S-definable languages coincide

Coding of Σ

Let $m \in \mathbb{N}$ be the smallest number such that $\|\Sigma\| \leq 2^m$.

W.l.o.g. assume that $\Sigma = \{0,1\}^m$, and let $X_1 \ldots X_p, x_{p+1}, \ldots x_m$

A word $w \in \Sigma^*$ induces an *interpretation* of $X_1 \ldots X_p, x_{p+1}, \ldots x_m$:

- $i \in I_w(X_j)$ iff the j-th element of w_i is 1, and
- $I_w(x_j) = i$ iff w_i has 1 on the j-th position and, for all $k \neq i$ w_k has 0 on the j-th position.

In the rest, let $\mathfrak{m}_w = \langle dom(w), \leq \rangle$ and ι_w be this interpretation.

Example

Example 1 *Let* $\Sigma = \{a, b, c, d\}$*, encoded as* $a = (00)$*,* $b = (01)$*,* $c = (10)$ *and* $d = (11)$*. Then the word abbaacdd induces the valuation* $X_1 = \{5, 6, 7\}$*,* $X_2 = \{1, 2, 6, 7\}$*.* $\square$

From Automata to Formulae

Let $A = \langle S, I, T, F \rangle$ with $S = \{s_1, \ldots, s_p\}$, and $\Sigma = \{0,1\}^m$.

Build $\Phi_A(X_1, \ldots, X_m)$ such that $\forall w \in \Sigma^* \,.\, w \in \mathcal{L}(A) \iff [\![\Phi_A]\!]^{\mathfrak{m}_w}_{\iota_w} = \text{true}$

Let $a \in \{0,1\}^m$. Let $\Phi_a(x, X_1, \ldots, X_m)$ be the conjunction of:

- $X_i(x)$ if the $a_i = 1$, and
- $\neg X_i(x)$ otherwise.

For all $w \in \Sigma^*$ we have $w \models \forall x \,.\, \bigvee_{a \in \Sigma} \Phi_a(x, \vec{X})$

Notice that $\Phi_a \wedge \Phi_b$ is unsatisfiable, for $a \neq b$.

Coding of S

Let $\{Y_0, \ldots, Y_p\}$ be set variables.

Y_i is the set of all positions labeled by A with state s_i during some run

$$\Phi_S(Y_1, \ldots, Y_p) \ : \ \forall z \ . \ \bigvee_{1 \leq i \leq p} Y_i(z) \ \wedge \ \bigwedge_{1 \leq i < j \leq p} \neg \exists z \ . \ Y_i(z) \wedge Y_j(z)$$

Coding of I

Every run starts from an initial state:

$$\Phi_I(Y_1, \ldots, Y_p) \ : \ \exists x \forall y \ . \ x \leq y \wedge \bigvee_{s_i \in I} Y_i(x)$$

Coding of T

Consider the transition $s_i \xrightarrow{a} s_j$:

$$\Phi_T(X_1, \ldots, X_m, Y_1, \ldots, Y_p) \;:\; \forall x \,.\, x \neq s(x) \wedge Y_i(x) \wedge \Phi_a(x, \vec{X}) \to \bigvee_{(s_i, a, s_j) \in T} Y_j(s(x))$$

Coding of F

The last state on the run is a final state:

$$\Phi_F(Y_1, \ldots, Y_p) \;:\; \exists x \forall y \,.\, y \leq x \wedge \bigvee_{s_i \in F} Y_i(x)$$

$$\Phi_A = \exists Y_1 \ldots \exists Y_p \,.\, \Phi_S \wedge \Phi_I \wedge \Phi_T \wedge \Phi_F$$

From Formulae to Automata

Let $\Phi(X_1, \ldots, X_p, x_{p+1}, \ldots, x_m)$ be a WS1S formula.

Build an automaton A_Φ such that $\forall w \in \Sigma^* \, . \, w \in \mathcal{L}(A) \iff [\![\Phi]\!]^{\mathfrak{m}_w}_{\iota_w} = \text{true}$

Let $\Phi(X_1, X_2, x_3, x_4)$ be:

1. $X_1(x_3)$
2. $x_3 \leq x_4$
3. $X_1 = X_2$

From Formulae to Automata

A_Φ is built by induction on the structure of Φ:

- for $\Phi = \phi_1 \wedge \phi_2$ we have $\mathcal{L}(A_\Phi) = \mathcal{L}(A_{\phi_1}) \cap \mathcal{L}(A_{\phi_2})$
- for $\Phi = \phi_1 \vee \phi_2$ we have $\mathcal{L}(A_\Phi) = \mathcal{L}(A_{\phi_1}) \cup \mathcal{L}(A_{\phi_2})$
- for $\Phi = \neg\phi$ we have $\mathcal{L}(A_\Phi) = \overline{\mathcal{L}(A_\phi)}$
- for $\Phi = \exists X_i \, . \, \phi$, we have $\mathcal{L}(A_\Phi) = pr_i(\mathcal{L}(A_\phi))$.

Consequences

Theorem 8 *A language $L \subseteq \Sigma^*$ is definable in WS1S iff it is recognizable.*

Corollary 1 *The SAT problem for WS1S is decidable.*

Lemma 3 *Any WS1S formula $\phi(X_1, \ldots, X_m)$ is equivalent to an WS1S formula of the form $\exists Y_1 \ldots \exists Y_p \,.\, \varphi$, where φ does not contain other set variables than $X_1, \ldots, X_m, Y_1, \ldots, Y_p$.*

Regular, Star Free and Aperiodic Languages

Regular Languages

Let Σ be an alphabet, and $X, Y \subseteq \Sigma^*$

$$
\begin{aligned}
XY &= \{xy \mid x \in X \text{ and } y \in Y\} \\
X^* &= \{x_1 \ldots x_n \mid n \geq 0,\ x_1, \ldots, x_n \in X\}
\end{aligned}
$$

The class of *regular languages* $\mathcal{R}(\Sigma)$ is the smallest class of languages $L \subseteq \Sigma^*$ such that:

- $\emptyset \in \mathcal{R}(\Sigma)$
- $\{\alpha\} \in \mathcal{R}(\Sigma)$, for all $\alpha \in \Sigma$
- if $X, Y \in \mathcal{R}(\Sigma)$ then $X \cup Y, XY, X^* \in \mathcal{R}(\Sigma)$

Regular, rational and recognizable languages

Theorem 9 (Kleene) *A set of finite words is recognizable if and only if it is regular.*

Proof in every textbook.

Rational = regular, in older books e.g.

Samuel Eilenberg. *Automata, Languages and Machines.* Academic Press, 1974

Star Free Languages

The class of *star-free languages* is the smallest class $SF(\Sigma)$ of languages $L \in \Sigma^*$ such that:

- $\emptyset, \{\epsilon\} \in SF(\Sigma)$ and $\{a\} \in SF(\Sigma)$ for all $a \in \Sigma$
- if $X, Y \in SF(\Sigma)$ then $X \cup Y, XY, \overline{X} \in SF(\Sigma)$

Example 2

- $\Sigma^* = \overline{\emptyset}$ *is star-free*
- *if* $B \subset \Sigma$, *then* $\Sigma^* B \Sigma^* = \bigcup_{b \in B} \Sigma^* b \Sigma^*$ *is star-free*
- *if* $B \subset \Sigma$, *then* $B^* = \overline{\Sigma^* \overline{B} \Sigma^*}$ *is star-free*
- *if* $\Sigma = \{a, b\}$, *then* $(ab)^* = \overline{b\Sigma^* \cup \Sigma^* a \cup \Sigma^* aa \Sigma^* \cup \Sigma^* bb \Sigma^*}$ *is star-free*

Aperiodic Languages

Definition 3 *A language $L \subseteq \Sigma^*$ is said to be* aperiodic *iff:*

$$\exists n_0 \forall n \geq n_0 \forall u, v, t \in \Sigma^* \ . \ uv^n t \in L \iff uv^{n+1} t \in L$$

n_0 is called the index *of L.*

Example 3 *0^*1^* is aperiodic. Let $n_0 = 2$. We have three cases:*

1. *$u, v \in 0^*$ and $t \in 0^*1^*$:*

$$\forall n \geq n_0 \ . \ uv^n t \in L$$

2. *$u \in 0^*$, $v \in 0^*1^*$ and $t \in 1^*$:*

$$\forall n \geq n_0 \ . \ uv^n t \notin L$$

3. *$u \in 0^*1^*$, $v \in 1^*$ and $t \in 1^*$:*

$$\forall n \geq n_0 \ . \ uv^n t \in L$$

Periodic Languages

Conversely, a language $L \subseteq \Sigma^*$ is said to be *periodic* iff:

$\forall n_0 \exists n \geq n_0 \exists u, v, t \in \Sigma^* \ . \ (uv^n t \notin L \wedge uv^{n+1} t \in L) \vee (uv^n t \in L \wedge uv^{n+1} t \notin L)$

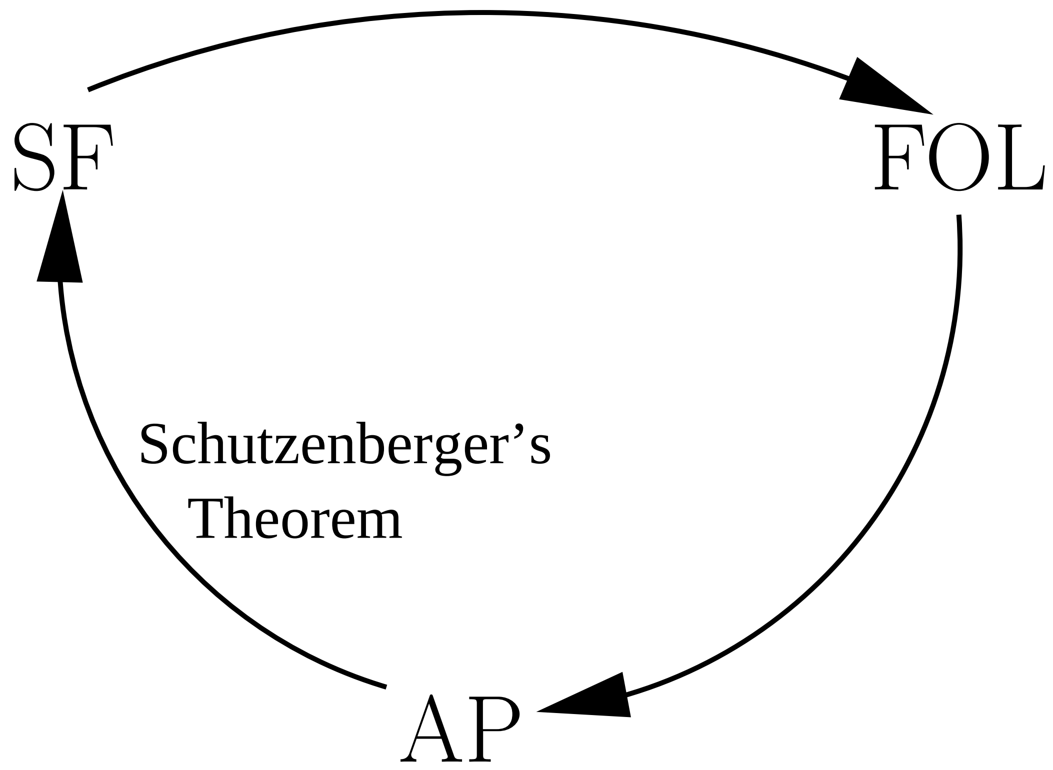
Example 4 *$(00)^*1$ is periodic.*

Given n_0 take the next even number $n \geq n_0$, $u = \epsilon$, $v = 0$ and $t = 1$. Then $uv^n t \in (00)^*1$ and $uv^{n+1} t \notin (00)^*1$. $\square$

Exercise 2 *Is $(00)^*1$ WS1S-definable ?*

Exercise 3 *Is the language $(ab)^*$ periodic or aperiodic ?*

The Big Picture



Subword Formulae

Let $w = a_0 a_1 \ldots a_{n-1}$ be a finite word, and $w(i,j) = a_i a_{i+1} \ldots a_{j-1}$ be a subword of w, $0 \leq i < n$ and $0 \leq j \leq n$, $i < j$.

Proposition 3 *For each FOL statement φ there exists a formula $\varphi(x,y)$ such that, for each $w \in \Sigma^*$ and each $0 \leq i < j \leq |w|$:*

$$w(i,j) \models \varphi \iff w \models \varphi(i,j)$$

By induction on the structure of φ:

$$\begin{array}{rcl}
(\neg\varphi)(x,y) & = & \neg(\varphi(x,y)) \\
(\varphi \wedge \psi)(x,y) & = & (\varphi(x,y)) \wedge (\psi(x,y)) \\
(\exists z.\varphi)(x,y) & = & \exists z \;.\; x \leq z \wedge z < y \wedge \varphi(x,y)
\end{array}$$

Star Free Languages are FOL-definable

We prove that for each $L \subseteq \Sigma^*$, $L \in SF(\Sigma)$ there exists an FOL sentence φ_L such that:

$$L = \{u \in \Sigma^* \mid u \models \varphi_L\}$$

By induction on the structure of L:

$$\emptyset = \{u \in \Sigma^* \mid u \models \bot\} \qquad \{a\} = \{u \in \Sigma^* \mid u \models p_a(0) \wedge len(1)\}$$

$$X \cup Y = \{u \in \Sigma^* \mid u \models \varphi_X \vee \varphi_Y\} \qquad \overline{X} = \{u \in \Sigma^* \mid u \models \neg\varphi_X\}$$

$$XY = \exists y \exists z \,.\, 0 \leq y < z \wedge \varphi_X(0, y) \wedge \varphi_Y(y, z) \wedge len(z)$$

where:

- $\varphi(i, j)$ is a formula s.t. $\forall 0 \leq i < j \leq |u| \,.\, u \models \varphi(i, j) \iff u(i, j) \models \varphi$
- $len(x) \equiv \forall y \,.\, s(y) \leq x$