

# Automata on Infinite Trees

# Büchi Automata on Infinite Trees

## Definition

A Büchi tree automaton over  $\Sigma$  is  $A = \langle S, I, T, F \rangle$ , where:

- $S$  is a finite set of *states*,
- $I \subseteq S$  is a set of *initial states*,
- $T : S \times \Sigma \rightarrow 2^{S \times S}$  is the *transition relation*,
- $F \subseteq S$  is the set of *final states*.

## Runs

A *run* of  $A$  over a tree  $t : \{0, 1\}^* \rightarrow \Sigma$  is a mapping  $\pi : \{0, 1\}^* \rightarrow S$  such that, for each position  $p \in \{0, 1\}^*$ , where  $q = \pi(p)$ , we have:

- if  $p = \epsilon$  then  $q \in I$ , and
- if  $q_i = \pi(pi)$ ,  $i = 0, 1$  then  $\langle q_0, q_1 \rangle \in T(q, t(p))$ .

If  $\pi$  is a *run* of  $A$  and  $\sigma$  is a *path* in  $t$ , let  $\pi|_\sigma$  denote the path in  $\pi$  corresponding to  $\sigma$ .

A run  $\pi$  is said to be *accepting*, if and only if for every path  $\sigma$  in  $t$  we have:

$$\text{inf}(\pi|_\sigma) \cap F \neq \emptyset$$

## Closure Properties

For every Büchi automaton  $A$  there exists a complete Büchi automaton  $A'$  such that  $\mathcal{L}(A) = \mathcal{L}(A')$ .

**Theorem 1** *The class of Büchi-recognizable tree languages is closed under union, intersection and projection.*

Let  $A_i = \langle S_i, I_i, T_i, F_i \rangle$ ,  $i = 1, 2$ , where  $S_1 \cap S_2 = \emptyset$ .

Let  $A_{\cup} = \langle S_1 \cup S_2, I_1 \cup I_2, T_1 \cup T_2, F_1 \cup F_2 \rangle$ .

## Closure Properties

Let  $A_{\cap} = \langle S, I, T, F \rangle$  where:

- $S = S_1 \times S_2 \times \{0, 1, 2\}$
- $I = I_1 \times I_2 \times \{1\}$
- for any  $s, s_1, s_2 \in S_1, s', s'_1, s'_2 \in S_2, a, b \in \{0, 1, 2\}$ :

$$\langle (s_1, s'_1, b), (s_2, s'_2, b) \rangle \in T((s, s', a), \sigma)$$

iff  $\langle s_1, s_2 \rangle \in T(s, \sigma), \langle s'_1, s'_2 \rangle \in T(s', \sigma)$  and:

1. if  $a = 0$  or ( $a = 1$  and  $s \notin F_1$ ), then  $b = 1$
  2. if ( $a = 1$  and  $s \in F_1$ ) or ( $a = 2$  and  $s \notin F_1$ ), then  $b = 2$
  3. if  $a = 2$  and  $s' \in F_2$ , then  $b = 0$
- $F = S \times S \times \{0\}$

## Emptiness of Büchi Tree Automata

Let  $A = \langle S, I, T, F \rangle$  be a Büchi tree automaton where  $F = \{s_1, \dots, s_m\}$ , and  $\pi : \{0, 1\}^* \rightarrow S$  be a **successful run** of  $A$  on the tree  $t \in \mathcal{T}(\Sigma)$ .

For any  $s \in S$ , and any  $u \in \{0, 1\}^*$  such that  $\pi(u) = s$ , let

$$d_u^\pi = \{w \in u \cdot \{0, 1\}^* \mid \forall v . u < v < w \Rightarrow \pi(v) \notin F\}$$

By König's lemma,  $d_u^\pi$  is finite for any  $u \in \{0, 1\}^*$ .

Let  $t_s^\pi$  be the restriction of  $t$  to  $d_u^\pi$ . Let

$$T_s = \{t_s^\pi \mid \pi \text{ is a successful run of } A \text{ on } t\}$$

## Emptiness of Büchi Tree Automata

If  $\vec{s} = \langle s_1, \dots, s_m \rangle$  are the final states of  $A$ :

$$\mathcal{L}(A) = \bigcup_{s_0 \in I} T_{s_0} \cdot_{\vec{s}} \langle T_{s_1}, \dots, T_{s_m} \rangle^{\omega \vec{s}}$$

Conversely, the expression above denotes a Büchi-recognizable tree language.

Let  $A = \langle S, I, T, F \rangle$  be a Büchi tree automaton. For each  $s \in S$  let  $T_s$  be the recognizable tree language defined above. Eliminate from  $S$  (and  $T$ ) all states  $s$  such that  $T_s = \emptyset$ , and let  $S'$  be the resulting set of states.

We claim that  $\mathcal{L}(A) \neq \emptyset \iff S' \cap I \neq \emptyset$ .



## The Complement Problem

Let  $\Sigma = \{a, b\}$ ,  $\mathcal{T}_0 = \{t \in \mathcal{T}^\omega(\Sigma) \mid \text{some path in } t \text{ has infinitely many } a\text{'s}\}$

$\mathcal{T}_0$  is Büchi recognizable.

Let  $A = \langle \{s_0, s_1, s_a, s_b\}, \{s_0\}, T, \{s_1, s_a\} \rangle$ , where  $T$  is defined by:

$$a(s_{0,a,b}) \rightarrow \{\langle s_1, s_a \rangle, \langle s_a, s_1 \rangle\}$$

$$b(s_{0,a,b}) \rightarrow \{\langle s_1, s_b \rangle, \langle s_b, s_1 \rangle\}$$

$$a(s_1) \rightarrow \{\langle s_1, s_1 \rangle\}$$

$$b(s_1) \rightarrow \{\langle s_1, s_1 \rangle\}$$

## The Complement Problem

Let  $\mathcal{T}_1 = \mathcal{T}^\omega(\Sigma) \setminus \mathcal{T}_0 = \{t \in \mathcal{T}^\omega(\Sigma) \mid \text{all paths in } t \text{ have finitely many } a\text{'s}\}$ .

We show that  $\mathcal{T}_1$  cannot be recognized by a Büchi tree automaton.

*Exercise 1*  $I = \{s_0, s_1\}$ ,  $F = \{s_1\}$  and

$$a(s_0) \rightarrow \langle s_0, s_0 \rangle$$

$$\langle s_0, s_1 \rangle$$

$$\langle s_1, s_0 \rangle$$

$$\langle s_1, s_1 \rangle$$

$$b(s_0) \rightarrow \langle s_0, s_0 \rangle$$

$$\langle s_0, s_1 \rangle$$

$$\langle s_1, s_0 \rangle$$

$$\langle s_1, s_1 \rangle$$

$$b(s_1) \rightarrow \langle s_1, s_1 \rangle$$

## The Complement Problem

Let  $T_n : \{0, 1\}^* \rightarrow \Sigma$  be the language of trees:

$$t_n(p) = \begin{cases} a & \text{if } p \in \{\epsilon, 1^{m_1}0, 1^{m_1}01^{m_2}0, \dots, 1^{m_1}01^{m_2}0 \dots 1^{m_n}0 \mid m_1, \dots, m_n \in \mathbb{N}\} \\ b & \text{otherwise} \end{cases}$$

Obviously,  $T_n \subset \mathcal{T}_1$ , for all  $n \in \mathbb{N}$ .

Suppose there exists a Büchi automaton  $A = \langle S, I, T, F \rangle$  with  $k$  states, s.t.  $\mathcal{L}(A) = \mathcal{T}_1$ . Let  $\pi$  be the accepting run of  $A$  over  $t_{k+1}$ . Then there exist:

- $m_1 > 0$  such that  $\pi(1^{m_1}) = s_1 \in F$
- $m_2 > 0$  such that  $\pi(1^{m_1}01^{m_2}) = s_2 \in F$
- ...

There exists a path  $\sigma$  in  $t_m$  and  $u < v < w < \sigma$ , such that

$\pi(u) = \pi(w) = s \in F$  and  $t_m(v) = a$ . Then  $\pi = r_1 \cdot_s r_2 \cdot_s r_3$ , and  $r_1 \cdot_s r_2^{\omega s}$  is a successful run on  $q_1 \cdot q_2^\omega$ , which contains a path with infinitely many  $a$ .

# Muller Automata on Infinite Trees

## Definition

A **Muller** tree automaton  $\Sigma$  is  $A = \langle S, I, T, \mathcal{F} \rangle$ , where:

- $S$  is a finite set of *states*,
- $I \subseteq S$  is a set of *initial states*,
- $T : S \times \Sigma \rightarrow 2^{S \times S}$  is the *transition function*,
- $\mathcal{F} \subseteq 2^S$ , is the set of *accepting sets*.

A run  $\pi$  of  $A$  over  $t$  is said to be *accepting*, iff for every path  $\sigma$  in  $t$ :

$$\text{inf}(\pi|_{\sigma}) \in \mathcal{F}$$

## Closure Properties

The class of Muller-recognizable tree languages is closed under union and intersection.

For **union**, the proof is exactly as in the case of Büchi automata. For  $A_U$ , the set of accepting sets is the union of the sets  $\mathcal{F}_i$ ,  $i = 1, 2$ .

For **intersection**, let  $A_\cap = \langle S_1 \times S_2, I_1 \times I_2, T, \mathcal{F} \rangle$ , where:

- $\langle (s_1, s'_1), (s_2, s'_2) \rangle \in T((s, s'), \sigma)$  iff  $\langle s_1, s_2 \rangle \in T(s, \sigma)$  and  $\langle s'_1, s'_2 \rangle \in T(s', \sigma)$ , and
- $\mathcal{F} = \{G \in S_1 \times S_2 \mid pr_1(G) \in \mathcal{F}_1 \text{ and } pr_2(G) \in \mathcal{F}_2\}$ , where:
  - $pr_1(G) = \{s \in S_1 \mid \exists s' . (s, s') \in G\}$ , and
  - $pr_2(G) = \{s \in S_2 \mid \exists s' . (s', s) \in G\}$ .

# Rabin Automata on Infinite Trees

## Definition

A **Rabin** tree automaton  $\Sigma$  is  $A = \langle S, I, T, \Omega \rangle$ , where:

- $S$  is a finite set of *states*,
- $I \subseteq S$  is a set of *initial states*,
- $T : S \times \Sigma \rightarrow 2^{S \times S}$  is the *transition function*,
- $\Omega = \{ \langle N_1, P_1 \rangle, \dots, \langle P_n, N_n \rangle \}$  is the set of *accepting pairs*.

A run  $\pi$  of  $A$  over  $t$  is said to be *accepting*, if and only if for every path  $\sigma$  in  $t$  there exists a pair  $\langle N_i, P_i \rangle \in \Omega$  such that:

$$\inf(\pi|_{\sigma}) \cap N_i = \emptyset \text{ and } \inf(\pi|_{\sigma}) \cap P_i \neq \emptyset$$



## Büchi, Muller and Rabin

For every Büchi tree automaton  $A$  there exists a Muller tree automaton  $B$ , such that  $\mathcal{L}(A) = \mathcal{L}(B)$ , but not viceversa.

For every Muller tree automaton  $A$  there exists a Rabin tree automaton  $B$ , such that  $\mathcal{L}(A) = \mathcal{L}(B)$ , and viceversa.

## From Büchi to Muller

For each (nondeterministic) Büchi automaton  $A$  there exists a (nondeterministic) Muller automaton  $B$  such that  $\mathcal{L}(A) = \mathcal{L}(B)$

Let  $A = \langle S, I, T, F \rangle$  be a Büchi automaton.

Define  $B = \langle S, I, T, \{G \in 2^S \mid G \cap F \neq \emptyset\} \rangle$

Allowing Muller automata to be nondeterministic is essential here.

## From Rabin to Muller

Given a Rabin automaton  $A = \langle S, I, T, \Omega \rangle$ , such that

$$\Omega = \{ \langle N_1, P_1 \rangle, \dots, \langle N_k, P_k \rangle \}$$

let  $B = \langle S, I, T, \mathcal{F} \rangle$  be the Muller automaton, where

$$\mathcal{F} = \{ F \subseteq S \mid F \cap N_i = \emptyset \text{ and } F \cap P_i \neq \emptyset \text{ for some } 1 \leq i \leq k \}$$

## From Muller to Rabin

Given a Muller automaton  $A = \langle S, I, T, \mathcal{F} \rangle$ , there exists a Rabin automaton  $B$  such that  $\mathcal{L}(A) = \mathcal{L}(B)$

Let  $\mathcal{F} = \{Q_1, \dots, Q_k\}$

Let  $B = \langle S', I', T', \Omega' \rangle$  where:

- $S' = 2^{Q_1} \times \dots \times 2^{Q_k} \times S$
- $I' = \{ \langle \emptyset, \dots, \emptyset, s_0 \rangle \mid s_0 \in I \}$

## From Muller to Rabin

- $T'(\langle S_1, \dots, S_k, s \rangle, a) = (\langle S'_1, \dots, S'_k, s' \rangle, \langle S''_1, \dots, S''_k, s'' \rangle)$  where:
  - $T(s, a) = (s', s'')$
  - for all  $1 \leq i \leq k$ :

$$S'_i = S''_i = \begin{cases} \emptyset & , \text{ if } S_i \cup \{s\} = Q_i \\ (S_i \cup \{s\}) \cap Q_i & , \text{ otherwise} \end{cases}$$

- $P_i = \{\langle S_1, \dots, S_i, \dots, S_k, s \rangle \mid S_i = Q_i\}$ ,  $1 \leq i \leq k$
- $N_i = \{\langle S_1, \dots, S_i, \dots, S_k, s \rangle \mid s \notin Q_i\}$ ,  $1 \leq i \leq k$

## The Rabin Complementation Theorem

**Theorem 2 (Rabin '69)** *The class of Rabin-recognizable tree languages is closed under complement.*

The class of Rabin-recognizable tree languages is closed under union and intersection, because Muller-recognizable languages are.

## Emptiness of Rabin Automata

Given an alphabet  $\Sigma$ , an infinite tree  $t \in \mathcal{T}^\omega(\Sigma)$  is said to be *regular* if there are only finitely many distinct subtrees  $t_u$  of  $t$ , where  $u \in \{0, 1\}^*$ .

**Example 1** *The infinite binary tree  $f(g(f(\dots), f(\dots)), g(f(\dots), f(\dots)))$  is regular.  $\square$*

### **Theorem 3 (Rabin '72)**

1. *Any non-empty Rabin-recognizable set of trees contains a regular tree.*
2. *The emptiness problem for Rabin tree automata is decidable.*

## Reduction to empty alphabet

Let  $A = \langle S, I, T, \Omega \rangle$  be a Rabin tree automaton over  $\Sigma$ , such that  $\mathcal{L}(A) \neq \emptyset$ , where  $\Omega = \{\langle N_1, P_1 \rangle, \dots, \langle N_n, P_n \rangle\}$ .

Let  $A' = \langle S \times \Sigma, I \times \Sigma, T', \Omega' \rangle$ , where:

- $\langle (s_1, \sigma_1), (s_2, \sigma_2) \rangle \in T'((s, \sigma))$  iff  $\langle s_1, s_2 \rangle \in T(s, \sigma)$ , and  $\sigma_1, \sigma_2 \in \Sigma$ .
- $\Omega' = \{\langle N_1 \times \Sigma, P_1 \times \Sigma \rangle, \dots, \langle N_n \times \Sigma, P_n \times \Sigma \rangle\}$ .

The successful runs of  $A'$  are pairs  $(\pi, t)$ , where  $t \in \mathcal{L}(A)$ , and  $\pi$  is a successful run of  $A$  on  $t$ .



## Regular successful runs

For any Rabin tree automaton  $A$ , there exists a Rabin tree automaton  $A'$  **with one initial state** such that  $\mathcal{L}(A) = \mathcal{L}(A')$ .

Consider a Rabin tree automaton  $A = \langle S, s_0, T, \Omega \rangle$  over the empty alphabet, and let  $\pi$  be a successful run of  $A$ .

**Claim 1** *If  $A$  has a successful run,  $A$  has also a **regular** successful run.*

A state  $s \in S$  is said to be **live** if  $s \neq s_0$  and  $\langle s_1, s_2 \rangle \in T(s)$  for some  $s_1, s_2 \in S$ , where either  $s_1 \neq s$  or  $s_2 \neq s$ .

By induction on  $n =$  the number of live states in  $A$ .

## Regular successful runs

Base case  $n = 0$ :  $\pi(\epsilon) = s_0$  and  $\pi(p) = s$ , for all  $p \in \text{dom}(\pi)$ , and  $s \in S$  is non-live.

Inductive step  $n > 0$ :

**Case 1** If some live state in  $A$  is missing on  $\pi$ , apply the induction hypothesis.

**Case 2** All live states of  $A$  appear on  $\pi$ , and there is a position  $u \in \{0, 1\}^*$  such that  $\pi(u) = s$  is live, but some live state  $s'$  does not appear in  $\pi_u$ .

Let  $\pi_1 = \pi \setminus \pi_u$  and  $\pi_2 = \pi_u$ . Both  $\pi_1$  and  $\pi_2$  are runs of automata with  $n - 1$  live states, hence there exists successful regular runs  $\pi'_1$  and  $\pi'_2$  of these automata. The desired run is  $\pi'_1 \cdot_s \pi'_2$ .

## Regular successful runs

**Case 3** All live states appear in any subtree of  $\pi$ . Let  $\sigma$  be a path in  $\pi$  consisting of all the live states appearing again and again, and only of the live states, with the exception of  $\pi(\epsilon)$ . **Q: Why does  $\sigma$  exist?**

There exists  $\langle N, P \rangle \in \Omega$ , such that  $\text{inf}(\sigma) \cap N = \emptyset$  and  $\text{inf}(\sigma) \cap P \neq \emptyset$ .

Then  $N$  contains only non-live states.

Let  $s \in \text{inf}(\sigma) \cap P$  and  $u, v$  be the 1<sup>st</sup> and 2<sup>nd</sup> positions such that  $\sigma(u) = \sigma(v) = s$ .

Let  $\pi_1 = \pi \setminus \pi_u$  and  $\pi_2 = \pi_u \setminus \pi_v$ . Both  $\pi_1$  and  $\pi_2$  are runs of automata with  $n - 1$  live states, hence there exists successful regular runs  $\pi'_1$  and  $\pi'_2$  of these automata. The desired run is  $\pi'_1 \cdot_s \pi'^{\omega s}_2$ .

## The Emptiness Problem

Let  $A$  be an input-free Rabin tree automaton with  $n$  live states.

We derive  $A_{n-1}, A_{n-2}, \dots, A_0$  from  $A$ , having  $n - 1, n - 2, \dots, 0$  live states.

If  $A$  has a successful run, then it has a regular run, composed of runs of  $A_{n-1}, A_{n-2}, \dots, A_0$ .

So it is enough to check emptiness of  $A_{n-1}, A_{n-2}, \dots, A_0$ .

# Rabin Automata, SkS and $S\omega S$

## Defining infinite paths

We say that a set of positions  $X$  is **linear** iff the following holds:

$$\mathit{linear}(X) : (\forall x, y . X(x) \wedge X(y) \rightarrow x \leq y \vee y \leq x)$$

$X$  is a **path** iff:

$$\mathit{path}(X) : \mathit{linear}(X) \wedge \forall Y . \mathit{linear}(Y) \wedge X \subseteq Y \rightarrow X = Y$$

## From Automata to Formulae

Let  $A = \langle S, I, T, \Omega \rangle$  be a Rabin tree automaton, where  $S = \{s_1, \dots, s_p\}$ .

Let  $\vec{Y} = \{Y_1, \dots, Y_p\}$  be set variables.

If  $X$  denotes a path, state  $i$  appears infinitely often in  $X$  iff:

$$\text{inf}_i(X) : \forall x . X(x) \rightarrow \exists y . x \leq y \wedge X(y) \wedge Y_i(y)$$

The formula expressing the accepting condition is:

$$\Phi_\Omega(\vec{Y}) : \forall X . \text{path}(X) \rightarrow \bigvee_{\langle N, P \rangle \in \Omega} \left( \bigwedge_{s_i \in N} \neg \text{inf}_i(X) \wedge \bigvee_{s_i \in P} \text{inf}_i(X) \right)$$

## Decidability of S2S

**Theorem 4** *Given an alphabet  $\Sigma$ , a tree language  $L \subseteq \mathcal{T}^\omega(\Sigma)$  is definable in S2S iff it is recognizable.*

**Corollary 1** *The SAT problem for S2S is decidable.*



## Obtaining Decidability Results by Reduction

Suppose we have a logic  $\mathcal{L}$  interpreted over the domain  $\mathcal{D}$ , such that the following problem is decidable:

for each formula  $\varphi$  of  $\mathcal{L}$  there exists  $\mathfrak{m} \in \mathcal{D}$  such that  $\mathfrak{m} \models \varphi$

Then we can prove the same thing for another logic  $\mathcal{L}'$  interpreted over  $\mathcal{D}'$  iff there exists functions  $\Delta : \mathcal{D}' \rightarrow \mathcal{D}$  and  $\Lambda : \mathcal{L}' \rightarrow \mathcal{L}$  such that for all  $\mathfrak{m}' \in \mathcal{D}'$  and  $\varphi' \in \mathcal{L}'$  we have:

$$\mathfrak{m}' \models \varphi' \iff \Delta(\mathfrak{m}') \models \Lambda(\varphi')$$

## Decidability of $S\omega S$

Every tree  $t : \mathbb{N}^* \rightarrow \Sigma$  can be encoded as  $t' : \{0, 1\}^* \rightarrow \Sigma$ .

Let  $D = \{\epsilon\} \cup \{1^{n_1+1}01^{n_2+1}0 \dots 1^{n_k+1}0 \mid k \geq 1, n_i \in \mathbb{N}, 1 \leq i \leq k\}$ .

Embedding the domain of  $S\omega S$  into  $S2S$ :

$$D(x) \quad : \quad \exists z \forall y . z \leq y \wedge x = z \quad \vee \quad \forall y . s_0(y) \leq x \rightarrow \exists y' . y = s_1(y')$$

## Decidability of $S\omega S$

If  $p = 1^{n_1+1}01^{n_2+1}0 \dots 1^{n_k+1}0$ , let

$$f_i(p) = p \cdot 1^{i+1}0 = 1^{n_1+1}01^{n_2+1}0 \dots 1^{n_k+1}01^{i+1}0$$

$$x \preceq_D y \quad : \quad D(x) \wedge D(y) \wedge x \preceq y$$

Define the relation  $x \leq_D^{\exists} y$  iff  $x \in D$  and  $y = x \cdot 1^{n+1}0$ , for some  $n \in \mathbb{N}$ :

$$x \leq_D^{\exists} y \quad : \quad \exists z . y = s_0(z) \wedge \forall z' . x \leq z \wedge z' < z \rightarrow s_1(z') \leq y$$

Define  $f_0, f_1, f_2, \dots$  by induction:

$$f_0(x) = y \quad : \quad D(x) \wedge D(y) \wedge y = s_0(x)$$

$$f_{i+1}(x) = y \quad : \quad D(x) \wedge D(y) \wedge x \leq_D^{\exists} y \wedge \forall z . x \leq_D^{\exists} z \wedge \\ \bigwedge_{0 \leq k \leq i} z \neq f_k(x) \rightarrow y \preceq_D z$$