I. Recognisability by Semigroups and Monoids

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Semigroups

$(S, \cdot)$

$$a, b \in S \Rightarrow a \cdot b \in S$$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \qquad abc \overset{\text{def}}{=} a \cdot b \cdot c$$

Example 1 $(\{0,1,...,n-1\}, +_n)$ $\qquad x +_n y \overset{\text{def}}{=} x+y \bmod n$

Example 2 $(2^{A \times A}, \circ)$ $\qquad P \circ Q \overset{\text{def}}{=} \{(x,y) \mid \exists z \in A . (x,z) \in P \text{ and } (z,y) \in Q\}$

Example 3 $(\Sigma^+, \cdot)$ $\qquad u \cdot v \overset{\text{def}}{=} uv$

Semigroup Homomorphism

$h : (S, \cdot) \to (T, \circ)$ $h(a \cdot b) = h(a) \circ h(b)$

$S^+ \overset{\text{def}}{=}$ set of non-empty sequences of elements from S

$h^+(a_1, a_2, ..., a_n) \overset{\text{def}}{=} h(a_1), h(a_2), ..., h(a_n)$

$$
\begin{array}{ccc}
S^+ & \xrightarrow{\quad h^+ \quad} & T^+ \\
\Big\downarrow {\scriptstyle \cdot} & & \Big\downarrow {\scriptstyle \circ} \\
S & \xrightarrow{\quad h \quad} & T
\end{array}
$$

$h(a_1 \cdot a_2 \cdot ... \cdot a_n) \; = \; h(a_1) \circ h(a_2) \circ ... \circ h(a_n)$

Monoids

$$a, b \in S \Rightarrow a \cdot b \in S$$

$$(S, 1, \cdot) \qquad (a \cdot b) \cdot c = a \cdot (b \cdot c) \qquad abc \stackrel{\text{def}}{=} a \cdot b \cdot c$$

$$1 \cdot a = a \cdot 1 = a$$

Example 1 $\quad (\{0,1,...,n-1\}, 0, +_n) \qquad x +_n y \stackrel{\text{def}}{=} x+y \bmod n$

Example 2 $\quad (2^{A \times A}, Id_A, \circ) \qquad\qquad P \circ Q \stackrel{\text{def}}{=} \{(x,y) \mid \exists z \in A . (x,z) \in P \text{ and } (z,y) \in Q\}$

$$Id_A \stackrel{\text{def}}{=} \{(x,x) \mid x \in A\}$$

Example 3 $\quad (\Sigma^*, \varepsilon, \cdot) \qquad\qquad u \cdot v \stackrel{\text{def}}{=} uv$

Monoid Homomorphism
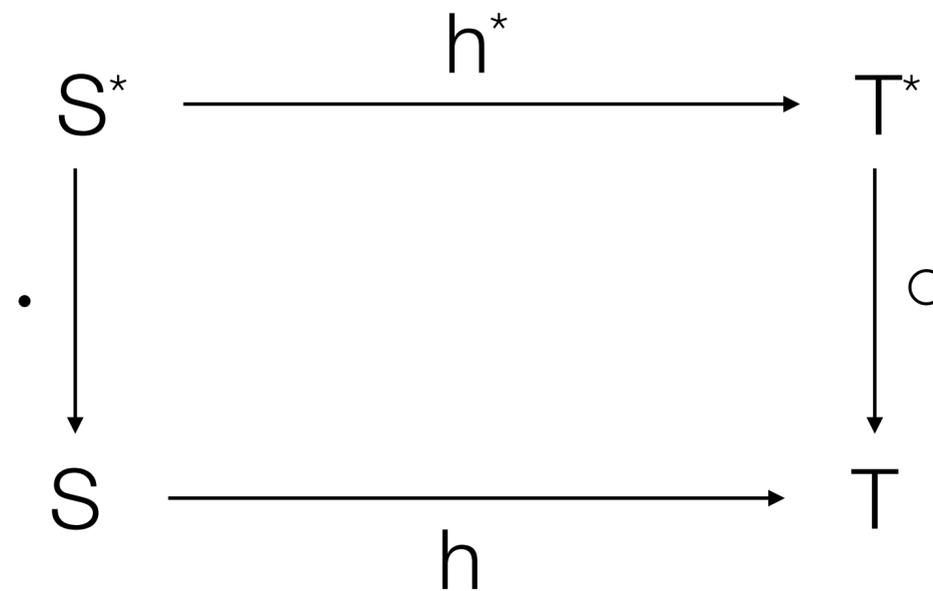
$h : (S, 1_S, \cdot) \rightarrow (T, 1_T, \circ)$

$h(a \cdot b) = h(a) \circ h(b)$

$\textcolor{red}{h(1_S) = 1_T}$

$S^* \stackrel{\text{def}}{=}$ set of sequences of elements from S

$h^*(a_1, a_2, ..., a_n) \stackrel{\text{def}}{=} h(a_1), h(a_2), ..., h(a_n)$

$\textcolor{red}{h^*(\varepsilon) \stackrel{\text{def}}{=} \varepsilon}$

$$
\begin{array}{ccc}
S^* & \xrightarrow{\ h^*\ } & T^* \\
\Big\downarrow {\scriptstyle \cdot} & & \Big\downarrow {\scriptstyle \circ} \\
S & \xrightarrow{\ h\ } & T
\end{array}
$$

$h(a_1 \cdot a_2 \cdot ... \cdot a_n) \;=\; h(a_1) \circ h(a_2) \circ ... \circ h(a_n)$

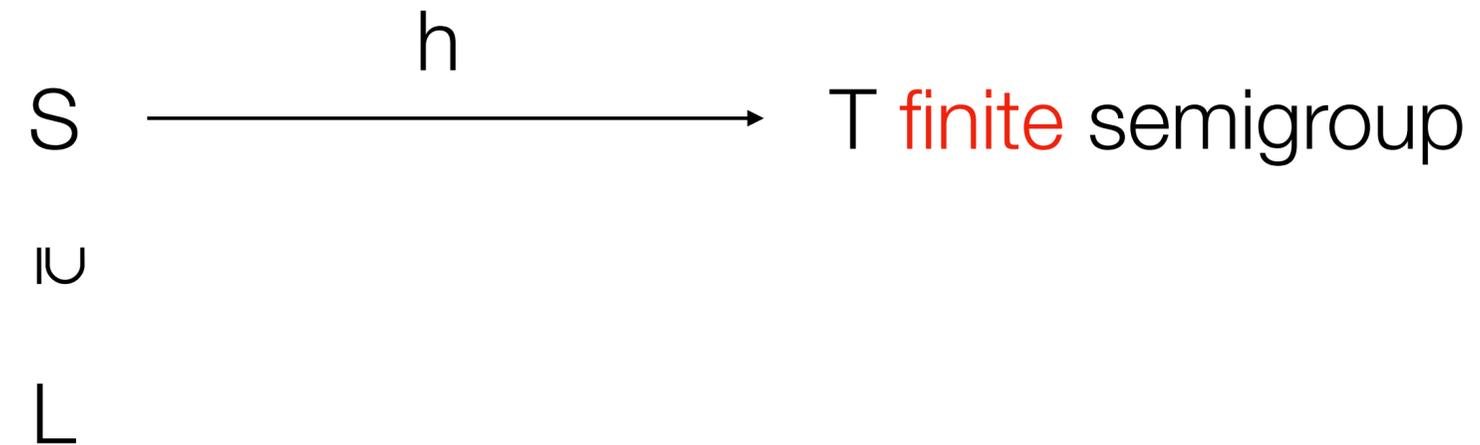# Recognisability
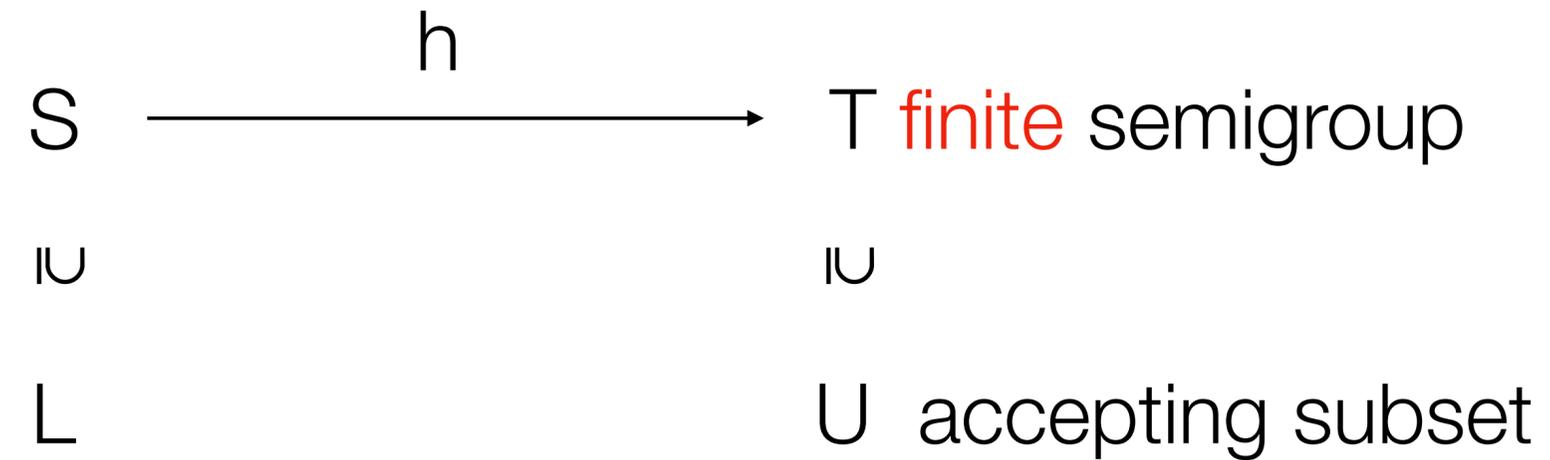
(S, ·) semigroup (same definition for S monoid)

S

S

L

Recognisability

(S, ·) semigroup (same definition for S monoid)

$$S \xrightarrow{\quad h \quad} T \text{ finite semigroup}$$

ıυ

L

Recognisability

(S, ·) semigroup (same definition for S monoid)

$$S \xrightarrow{\ h\ } T \text{ finite semigroup}$$

L

U accepting subset

Recognisability

(S, ·) semigroup (same definition for S monoid)

$$S \xrightarrow{\quad h \quad} T \text{ finite semigroup}$$

$$\text{IU} \qquad\qquad \text{IU}$$

$$L \quad = \quad h^{-1}(U) \text{ accepting subset}$$

Recognisability

(S, ·) semigroup (same definition for S monoid)

$$S \xrightarrow{\ h\ } T \text{ finite semigroup}$$

$$\cup \qquad\qquad \cup$$

$$L \qquad = \qquad h^{-1}(U) \text{ accepting subset}$$

Example 1   The set of odd (even) numbers is recognisable in $(\mathbb{N}, +)$

Recognisable Word Languages

Theorem $L \subseteq \Sigma^*$ is recognisable in the monoid $(\Sigma^*, \varepsilon, \cdot)$

$$\Leftrightarrow$$

L is the language of a nondeterministic finite automaton $A = (Q, I, F, \rightarrow)$

"$\Rightarrow$"

$$\Sigma^* \xrightarrow{\quad h \quad} (Q, 1_Q, \circ) \text{ finite monoid}$$

$$\text{I}\cup \qquad\qquad\qquad \text{I}\cup$$

$$L \qquad = \qquad h^{-1}(\,F\,)$$

$$A = (Q, \{1_Q\}, F, \rightarrow) \qquad q \xrightarrow{\;a\;} q \circ h(a)$$

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"⇐"

$$\Sigma^* \xrightarrow{\;h(w)=\{\ (q,q')\ |\ q \xrightarrow{w} q'\}\;} (2^{Q \times Q}, \mathrm{Id}_Q, \circ)$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$L \qquad\qquad = \qquad h^{-1}(\ \{R \in 2^{Q \times Q} \mid \exists i \in I\ \exists f \in F\ .\ (i,f) \in R\}\ )$$

Syntactic Monoids

$(S, 1_S, \cdot)$ monoid $\sim \, \subseteq S \times S$ $a \sim b \Rightarrow sat \sim sbt$, for all $s, t \in S$ congruence

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$(S_{/\sim L}, [1_S]_{\sim L}, \cdot)$ $[a]_{\sim L} \cdot [b]_{\sim L} \overset{\text{def}}{=} [ab]_{\sim L}$ syntactic monoid

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Theorem [Myhill-Nerode]

$L \subseteq S$ is recognisable $\Leftrightarrow S_{/\sim L}$ is finite
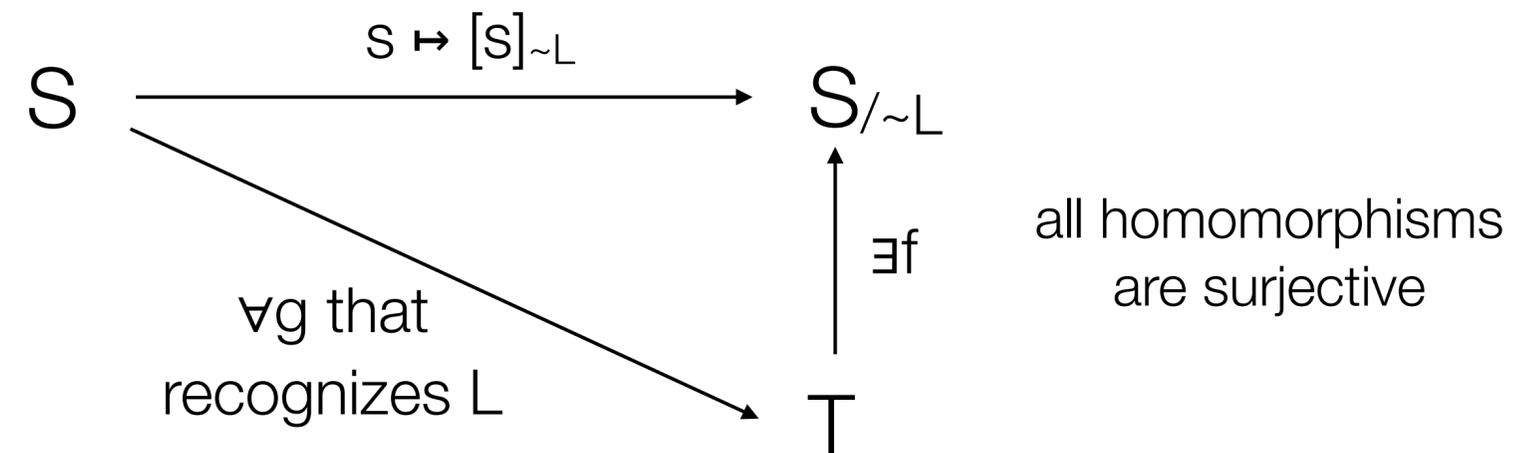
Syntactic Monoids

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Theorem [Myhill-Nerode]

$L \subseteq S$ is recognisable $\Leftrightarrow S_{/\sim L}$ is finite

$$S \xrightarrow{\;s \mapsto [s]_{\sim L}\;} S_{/\sim L}$$

$\forall g$ that recognizes L

$\exists f$

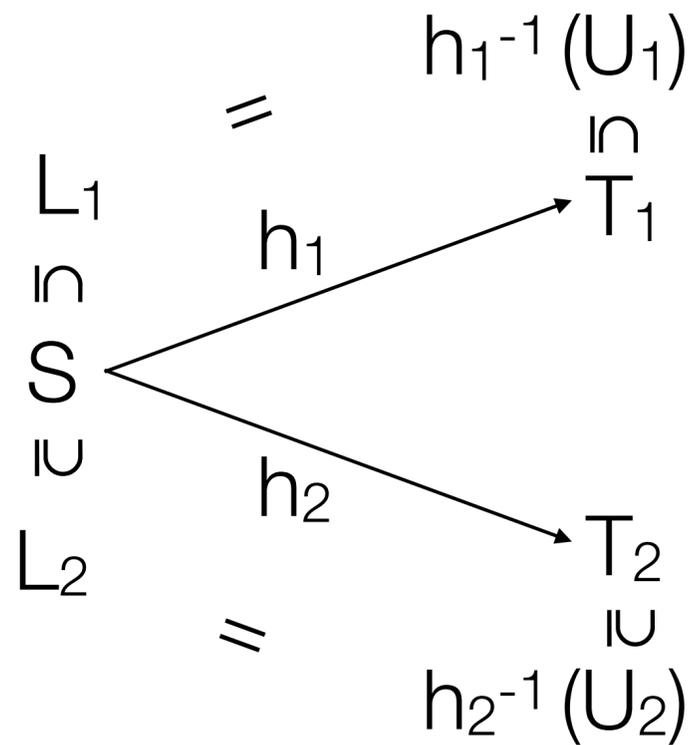all homomorphisms are surjective

T

Properties of Recognisable Sets

Recognisable sets are closed under boolean combinations

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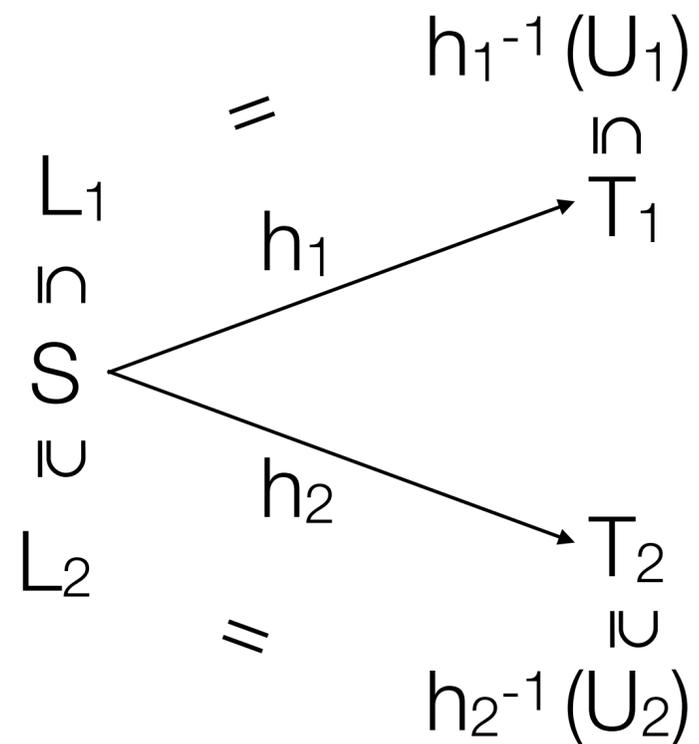
$$L_1 \stackrel{=}{} h_1^{-1}(U_1)$$

$$\cap$$

$$L_1 \qquad T_1$$

$$\cap \qquad h_1 \nearrow \qquad \cup$$

$$S \qquad \qquad$$

$$\cup \qquad h_2 \searrow$$

$$L_2 \qquad T_2$$

$$\stackrel{=}{} \qquad \cup$$

$$h_2^{-1}(U_2)$$

intersection

Properties of Recognisable Sets

Recognisable sets are closed under boolean combinations

$$h_1^{-1}(U_1)$$

$$L_1 \overset{=}{} \quad \begin{array}{c} \cap \\ T_1 \end{array}$$

$$\cap \qquad h_1$$

$$S$$

$$\cup \qquad h_2$$

$$L_2 \overset{=}{} \quad \begin{array}{c} T_2 \\ \cup \end{array}$$

$$h_2^{-1}(U_2)$$

$$(s_1,s_2) \otimes (t_1,t_2) \overset{\text{def}}{=} (s_1 t_1, s_2 t_2)$$

$$S \xrightarrow{\;\; h \overset{\text{def}}{=} (h_1, h_2) \;\;} (T_1 {\times} T_2, (1_{T_1}, 1_{T_2}), \otimes)$$

$$\cup \qquad\qquad\qquad\qquad\qquad \cup$$

$$L_1 \cap L_2 \qquad = \qquad h^{-1}(U_1 \times U_2)$$

intersection

Properties of Recognisable Sets

Recognisable sets are closed under boolean combinations

complement

$$S \xrightarrow{\ \ h\ \ } T$$

$$\cup \qquad\qquad\qquad \cup$$

$$L \qquad = \qquad h^{-1}(\cup)$$

Properties of Recognisable Sets

Recognisable sets are closed under boolean combinations

complement

$$S \xrightarrow{\;\;h\;\;} T$$

$$S \in \qquad\qquad\qquad T \in$$

$$L \qquad = \qquad h^{-1}(U)$$

$$S \xrightarrow{\;\;h\;\;} T$$

$$S \in \qquad\qquad\qquad T \in$$

$$S{\setminus}L \qquad = \qquad h^{-1}(T{\setminus}U)$$

Properties of Recognisable Sets

Recognisable sets are closed under inverse homomorphisms

$$S \xrightarrow{\ h\ } T$$

$$\cup \qquad\qquad\qquad \cup$$

$$L \qquad = \qquad h^{-1}(U)$$

Properties of Recognisable Sets

Recognisable sets are closed under inverse homomorphisms

$$S' \xrightarrow{\quad g \quad} S \xrightarrow{\quad h \quad} T$$

$$\cup \qquad\qquad \cup \qquad\qquad \cup$$

$$g^{-1}(\,L\,) = (h \circ g)^{-1}(U) \qquad L \qquad = \qquad h^{-1}(U\,)$$

Logic in General

$\mathcal{R} = r_1, r_2, \ldots$ signature of relation symbols

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$\forall x, \exists x$

quantification
over individuals

$\phi \wedge \psi, \phi \vee \psi, \neg \phi$
boolean
operations

$r(x_1,...,x_n)$

relations from
the signature

$x = y$
equality

First Order Logic (FO)

Logic in General

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equality

First Order Logic (FO)

$\forall X, \exists X$

quantification
over sets

$x \in X$

membership

Monadic Second Order Logic (MSO)

Logic in General

$\mathcal{R} = r_1, r_2, ...$ signature of relation symbols

$\mathcal{S} = \left(\mathcal{U}, r^{\mathcal{S}}_1, r^{\mathcal{S}}_2, ... \right)$ relational structure

universe   interpretation of
relation symbols

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$\mathcal{R} = r_1, r_2, ...$ signature of relation symbols

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universe   interpretation of relation symbols

$\mathcal{S}, u_1, ..., u_n, \mathcal{U}_1, ..., \mathcal{U}_m \vDash \phi \left(x_1, ..., x_n, X_1, ... X_m \right)$ the formula ϕ is true in $\mathcal{S}$ when

- x_i is mapped to the element u_i

- X_j is mapped to the set $\mathcal{U}_j$

Logic for Words (the precedence model)

$\mathcal{L} = \{a,b,c,...\}$

finite alphabet

$x \le y$

position x is

before position y

$a(x)$

position x has

letter a

Logic for Words (the precedence model)

$\mathscr{L} = \{a,b,c,...\}$ finite alphabet

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position x is   position x has

before position y letter a

$L(\phi) = \{\, w \in \mathscr{L} \mid w \vDash \phi \}$ language of a sentence

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$L \subseteq \mathcal{L}^*$ is FO (MSO) definable iff $L = L(\phi)$ for ϕ a FO (MSO) sentence

Some FO-definable Languages

$$x+1=y \overset{\text{def}}{=} \underbrace{x{\leq}y \wedge \neg x{=}y}_{x{<}y} \wedge \forall z \ . \ x{\leq}z \wedge z{\leq}y \Rightarrow x{=}z \vee z{=}y$$

Some FO-definable Languages

$x+1=y \overset{\text{def}}{=} x \leq y \wedge \neg x = y \wedge \forall z . x \leq z \wedge z \leq y \Rightarrow x = z \vee z = y$

$\underbrace{}_{x < y}$

$a*bc* \qquad \exists x . b(x) \wedge \forall y . y < x \Rightarrow a(x) \wedge \forall y . x < y \Rightarrow c(x)$

Some FO-definable Languages

$x+1=y \stackrel{\text{def}}{=} x{\leq}y \land \neg x{=}y \land \forall z . x{\leq}z \land z{\leq}y \Rightarrow x{=}z \lor z{=}y$

$\underbrace{\qquad\qquad\qquad}_{x<y}$

$a*bc*$ $\exists x . b(x) \land \forall y . y{<}x \Rightarrow a(x) \land \forall y . x{<}y \Rightarrow c(x)$

$first(x) \stackrel{\text{def}}{=} \forall y . x{\leq}y$ $last(x) \stackrel{\text{def}}{=} \forall y . y{\leq}x$

Some FO-definable Languages

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$first(x) \stackrel{\text{def}}{=} \forall y . x \leq y \qquad\qquad last(x) \stackrel{\text{def}}{=} \forall y . y \leq x$

$(ab)^+ \qquad \exists x . first(x) \wedge a(x) \wedge \forall y . a(y) \Rightarrow \neg last(y) \wedge b(y+1) \wedge \forall y . b(y) \wedge \neg last(y) \Rightarrow a(y+1)$

Some MSO-definable Languages

$$X \subseteq Y \stackrel{\text{def}}{=} \forall x \ . \ x \in X \Rightarrow x \in Y$$

$$X \subseteq a \stackrel{\text{def}}{=} \forall x \ . \ x \in X \Rightarrow a(x)$$

$$sing(X) \stackrel{\text{def}}{=} X \neq \varnothing \ \wedge \ \forall Y \ . \ Y \subseteq X \Rightarrow Y = \varnothing \ \vee \ Y = X$$

$$X \leqslant Y \stackrel{\text{def}}{=} \forall x \ \forall y \ . \ x \in X \ \wedge \ y \in Y \Rightarrow x \leqslant y$$

Some MSO-definable Languages

$(aa)^+$ all words of non-zero even length over the alphabet $\mathcal{L} = \{a\}$

$\exists X \, . \, \forall x \, . \, first(x) \Rightarrow x \in X \land last(x) \Rightarrow x \notin X \land \forall y \, . \, x = y+1 \Rightarrow (x \in X \Leftrightarrow y \notin X)$

all words having a on even positions over the alphabet $\mathcal{L} = \{a, b\}$

$\exists X \, . \, \forall x \, . \, first(x) \Rightarrow x \in X \land \forall y \forall z \, . \, y \in X \land z = y+1 \Rightarrow z \notin X \land X \subseteq a$

As we shall see later, these languages are not FO-definable ...

Logic for Words (the successor model)

$\mathscr{L} = \{a, b, c, ...\}$ finite alphabet

$x+1=y$ $a(x)$

position y is   position x has

the successor of x letter a

Makes a difference for FO-definability but not for MSO-definability

$$x \leqslant y \overset{\text{def}}{=} \forall X \, . \, \left(x \in X \wedge \left(\forall y \forall z \, . \, \underbrace{y \in X \wedge y+1=z \Rightarrow z \in X} \right) \right) \Rightarrow y \in X$$

closed under successors

MSO-definability = recognisability for words

Theorem [Trakhtenbrot-Büchi-Elgot]

Let $\mathscr{L}$ be a finite alphabet and L $\subseteq \mathscr{L}^*$ be a language.

L is recognisable $\Leftrightarrow$ L is MSO-definable.

"$\Rightarrow$"   L is recognisable $\Rightarrow$ L is the language of a finite automaton A with n states

$$\exists \mathcal{X}_1 \ldots \exists \mathcal{X}_n . \text{ the sets } \mathcal{X}_1 \ldots \mathcal{X}_n \text{ describe an accepting run of A}$$

$$\forall x . \bigvee_{1 \leq i \leq n} x \in \mathcal{X}_i \wedge \bigwedge_{1 \leq i < j \leq n} \neg \exists x . x \in \mathcal{X}_i \wedge x \in \mathcal{X}_j$$

$$\forall x . first(x) \Rightarrow \bigvee_{i \in \mathcal{I}} x \in \mathcal{X}_i \qquad \forall x . last(x) \Rightarrow \bigvee_{i \in \mathcal{F}} x \in \mathcal{X}_i$$

$$\forall x . \bigwedge_{a \in \mathscr{L}} \bigwedge_{1 \leq i \leq n} x \in \mathcal{X}_i \wedge a(x) \wedge \neg last(x) \Rightarrow \bigvee_{qi \xrightarrow{a} qj} x+1 \in \mathcal{X}_j$$

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"$\Leftarrow$" $X \subseteq Y$ $X \subseteq a$ $sing(X)$ $X \leqslant Y$

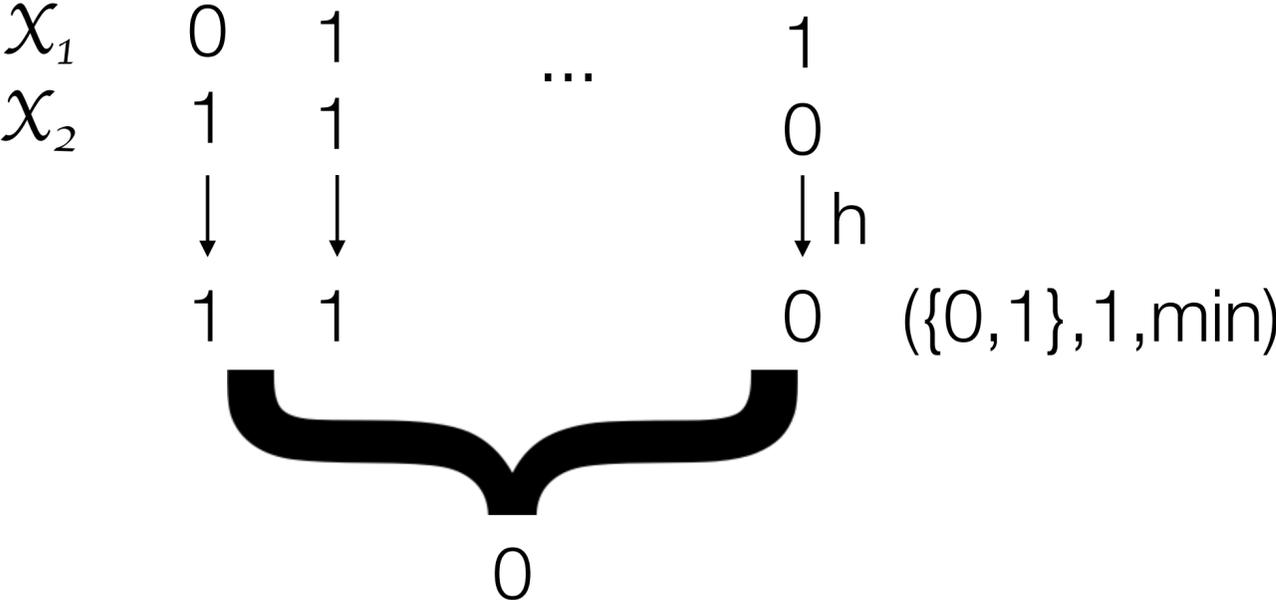each MSO formula can be written as $\phi(X_1 \dots X_n)$

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"$\Leftarrow$" $X \subseteq Y$ $X \subseteq a$ $sing(X)$ $X \leq Y$

each MSO formula can be written as $\phi(X_1 \dots X_n)$

the language of $\phi(X_1 \dots X_n)$ is the set of words w

over the alphabet $\mathscr{L} \times \{0,1\}^n : w, U_1, \dots, U_n \vDash \phi(X_1 \dots X_n)$

where $i \in U_j \Leftrightarrow$ the j-th bit on position i is 1

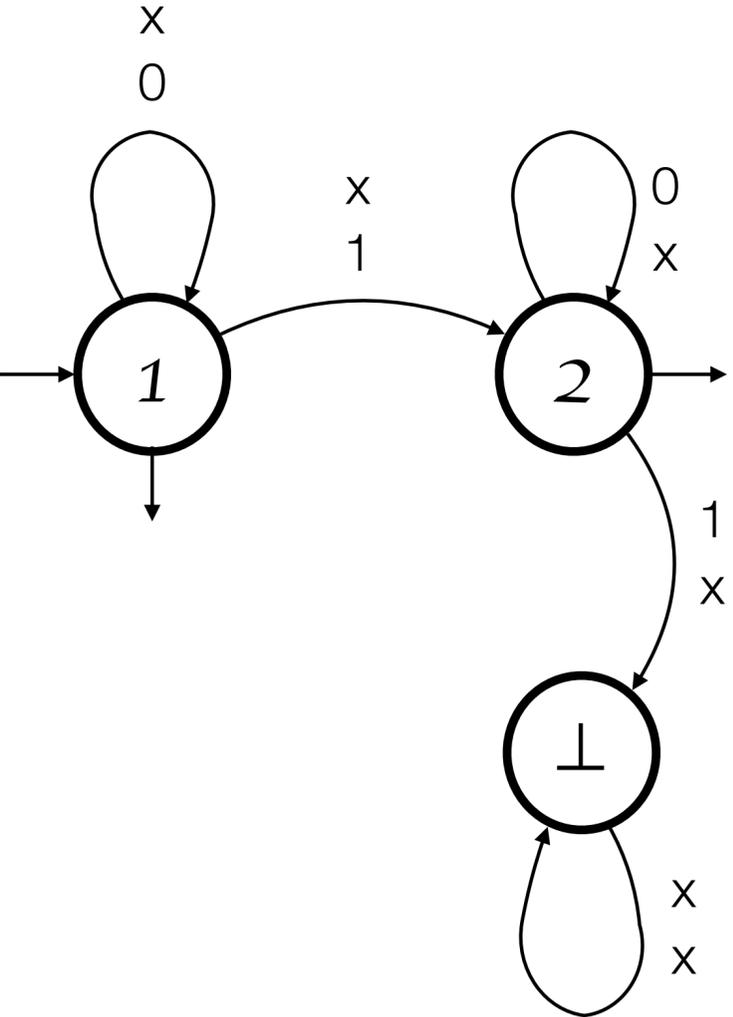|				
X_1	0	1	$\dots$	0
X_2	1	0	$\dots$	1
$\vdots$	$\vdots$	$\vdots$		$\vdots$
X_n	0	0	$\dots$	1
$\mathscr{L}$	a_1	a_2	$\dots$	a_m

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L is recognisable ⇔ L is MSO-definable.

"⇐" $\mathcal{X}_1 \subseteq \mathcal{X}_2$

$\mathcal{X}_1 \subseteq a$ similar construction

MSO-definability = recognisability for words

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"$\Leftarrow$" $\phi \wedge \phi, \phi \vee \phi, \neg \phi$ closure of recognizable sets under boolean operations

$$(\mathscr{E} \times \{0,1\}^{n-1})* \xleftarrow{\pi} (\mathscr{E} \times \{0,1\}^n)* \xrightarrow{h} M$$

$$\cup \qquad\qquad\qquad \cup$$

$$\phi(X_1 \ldots X_n) \quad = \quad h^{-1}(N)$$

$$(\mathscr{E} \times \{0,1\}^{n-1})* \xrightarrow{H(w)=\{h(v) \mid \pi(v)=w\}} 2^M$$

$$\cup \qquad\qquad\qquad \cup$$

$$\exists X_n . \phi(X_1 \ldots X_n) \quad = \quad H^{-1}(\{P \subseteq M \mid P \cap N \neq \varnothing\})$$