I. Recognisability by Semigroups and Monoids

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Semigroups

$(S, \cdot)$

$a, b \in S \Rightarrow a \cdot b \in S$

$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

$abc \stackrel{\text{def}}{=} a \cdot b \cdot c$

Example 1 $(\{0,1,...,n-1\}, +_n)$ $x +_n y \stackrel{\text{def}}{=} x+y \bmod n$

Example 2 $(2^{A\times A}, \circ)$ $P \circ Q \stackrel{\text{def}}{=} \{(x,y) \mid \exists z \in A \,.\, (x,z) \in P \text{ and } (z,y) \in Q\}$

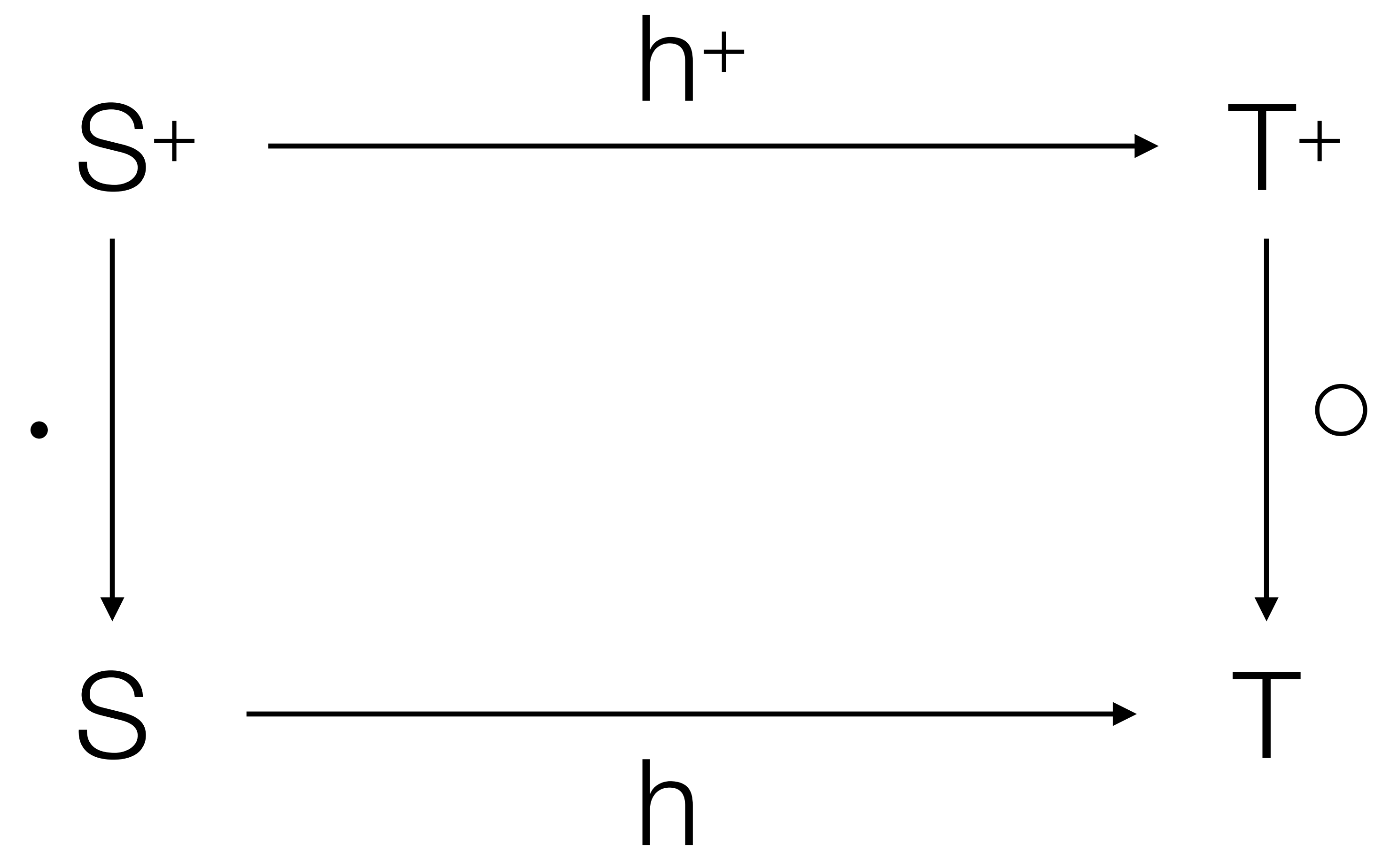
Example 3 $(\Sigma^+, \cdot)$ $u \cdot v \stackrel{\text{def}}{=} uv$

Semigroup Homomorphism

$h : (S, \cdot) \rightarrow (T, \circ)$ $\quad\quad$ $h(a \cdot b) = h(a) \circ h(b)$

$S^+ \stackrel{\text{def}}{=}$ set of non-empty sequences of elements from S

$h^+(a_1, a_2, ..., a_n) \stackrel{\text{def}}{=} h(a_1), h(a_2), ..., h(a_n)$



$h(a_1 \cdot a_2 \cdot ... \cdot a_n) = h(a_1) \circ h(a_2) \circ ... \circ h(a_n)$

Monoids

$$a, b \in S \Rightarrow a \cdot b \in S$$

$(S, 1, \cdot)$ $\quad$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ $\quad$ $abc \stackrel{def}{=} a \cdot b \cdot c$

$$1 \cdot a = a \cdot 1 = a$$

Example 1 $\quad$ $(\{0,1,\ldots,n-1\}, 0, +_n)$ $\quad$ $x +_n y \stackrel{def}{=} x+y \bmod n$

Example 2 $\quad$ $(2^{A \times A}, Id_A, \circ)$ $\quad$ $P \circ Q \stackrel{def}{=} \{(x,y) \mid \exists z \in A \,.\, (x,z) \in P \text{ and } (z,y) \in Q\}$

$Id_A \stackrel{def}{=} \{(x,x) \mid x \in A\}$

Example 3 $\quad$ $(\Sigma^*, \varepsilon, \cdot)$ $\quad$ $u \cdot v \stackrel{def}{=} uv$

Monoid Homomorphism

$h : (S, 1_S, \cdot\,) \rightarrow (T, 1_T, \circ)$

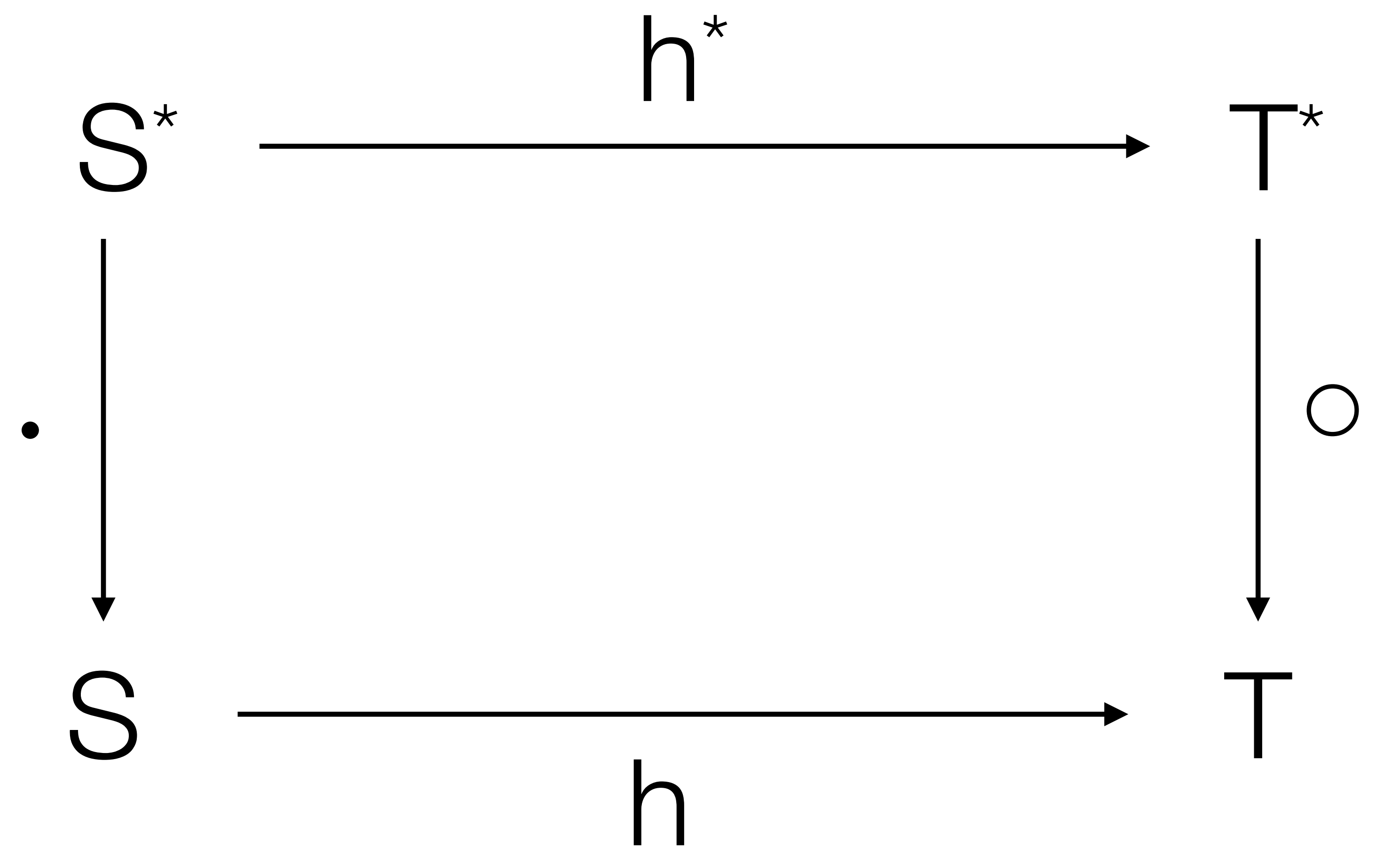
$h(a \cdot b) = h(a) \circ h(b)$

$h(1_S) = 1_T$

$S^* \overset{\text{def}}{=}$ set of sequences of elements from S

$h^*(a_1, a_2, ..., a_n) \overset{\text{def}}{=} h(a_1), h(a_2), ..., h(a_n)$

$h^*(\varepsilon) \overset{\text{def}}{=} \varepsilon$



$h(a_1 \cdot a_2 \cdot ... \cdot a_n) \;=\; h(a_1) \circ h(a_2) \circ ... \circ h(a_n)$

Recognisability

(S, ・) semigroup (same definition for S monoid)

$$S$$
$$\cup\!\!\text{I}$$
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$$\begin{array}{c} S \\ \cup\!\!| \\ L \end{array}$$

Recognisability

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$$
\begin{array}{ccc}
S & \xrightarrow{\quad h \quad} & T \text{ finite semigroup} \\
\cup\!| & & \cup\!| \\
L & & U \text{ accepting subset}
\end{array}
$$

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$$\begin{array}{ccc} S & \xrightarrow{\quad h \quad} & T \text{ finite semigroup} \\ \cup\!\!\text{I} & & \cup\!\!\text{I} \\ L & = & h^{-1}(U) \text{ accepting subset} \end{array}$$

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$$\begin{array}{ccc} S & \xrightarrow{\quad h \quad} & T \text{ finite semigroup} \\ \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\ L & = & h^{-1}(U) \text{ accepting subset} \end{array}$$

Example 1 The set of odd (even) numbers is recognisable in $(\mathbb{N}, +)$

Recognisable Word Languages

Theorem $L \subseteq \Sigma^*$ is recognisable in the monoid $(\Sigma^*, \varepsilon, \cdot)$

$\Leftrightarrow$

L is the language of a nondeterministic finite automaton $A = (Q, I, F, \rightarrow)$

"⇒"

$$\begin{array}{ccc} \Sigma^* & \xrightarrow{\quad h \quad} & (Q, 1_Q, \circ) \text{ finite monoid} \\ \cup\!\!| & & \cup\!\!| \\ L & = & h^{-1}(F) \end{array}$$

$A = (Q, \{1_Q\}, F, \rightarrow) \qquad q \overset{a}{\rightarrow} q \circ h(a)$

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$$\Sigma^* \xrightarrow{h(w)=\{ (q,q') \mid q \overset{w}{\rightarrow} q'\}} (2^{Q\times Q}, Id_Q, \circ)$$

$$\cup\!\!| \qquad\qquad\qquad \cup\!\!|$$

$$L \quad = \quad h^{-1}(\{R \in 2^{Q\times Q} \mid \exists i \in I\ \exists f \in F\ .\ (i,f) \in R\})$$

Syntactic Monoids

$(S, 1_S, \cdot)$ monoid $\sim \subseteq S \times S$ $a \sim b \Rightarrow sat \sim sbt$, for all $s, t \in S$ congruence

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$L \subseteq S$ is recognisable $\Leftrightarrow$ $S_{/\sim L}$ is finite

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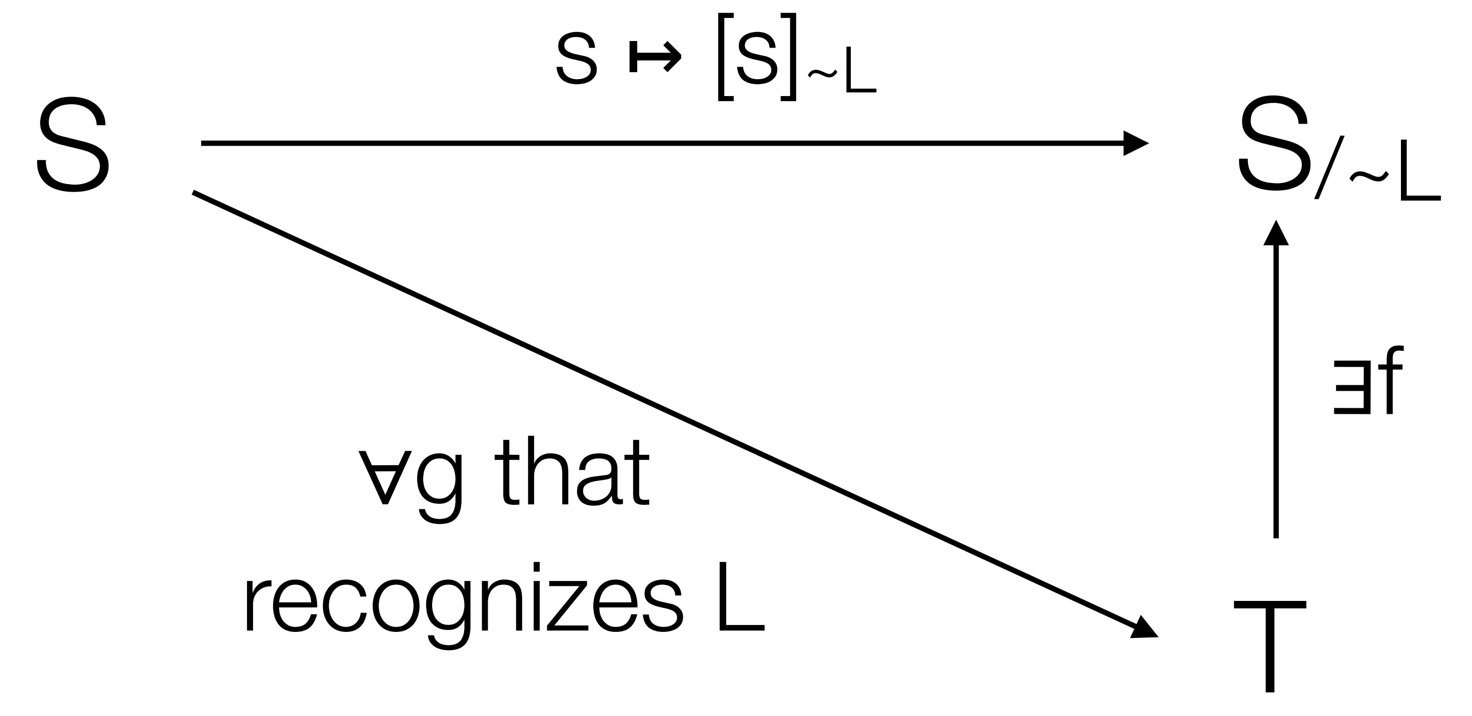

all homomorphisms
are surjective

Properties of Recognisable Sets

Recognisable sets are closed under boolean combinations

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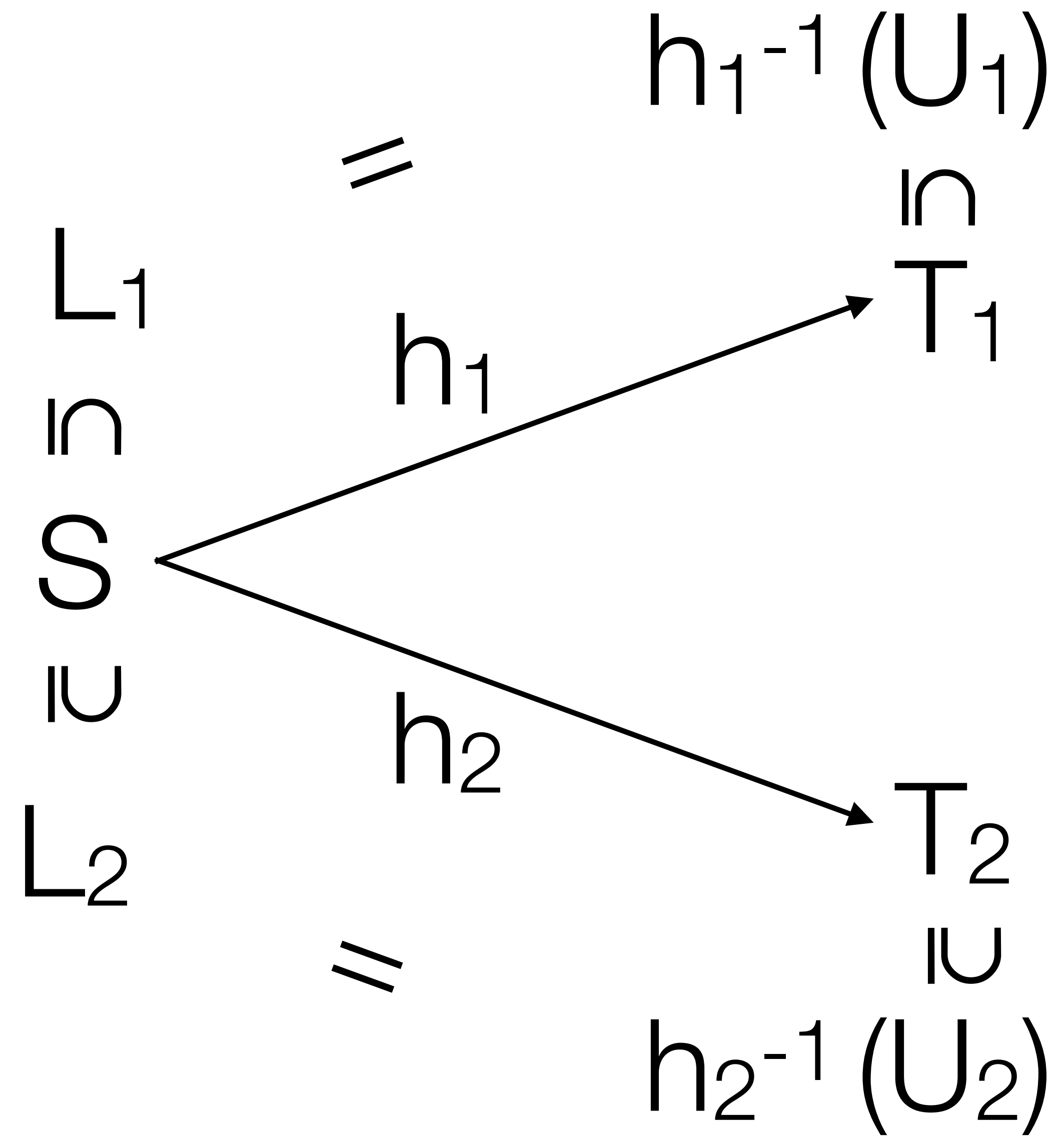
intersection



Properties of Recognisable Sets

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intersection

$$L_1 = h_1^{-1}(U_1), \quad L_1 \subseteq S, \quad U_1 \subseteq T_1, \quad S \xrightarrow{h_1} T_1$$

$$L_2 = h_2^{-1}(U_2), \quad L_2 \subseteq S, \quad U_2 \subseteq T_2, \quad S \xrightarrow{h_2} T_2$$

$$(s_1,s_2) \otimes (t_1,t_2) \stackrel{\text{def}}{=} (s_1 t_1,\ s_2 t_2)$$

$$S \xrightarrow{h \stackrel{\text{def}}{=} (h_1,\ h_2)} (T_1 x T_2,\ (1_{T1}, 1_{T2}),\ \otimes)$$

$$L_1 \cap L_2 \subseteq S, \qquad h^{-1}(U_1 \text{ x } U_2) \subseteq T_1 x T_2$$

$$L_1 \cap L_2 \quad = \quad h^{-1}(U_1 \text{ x } U_2)$$

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complement

$$\begin{array}{ccc} S & \xrightarrow{\quad h \quad} & T \\ \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\ L & = & h^{-1}(U) \end{array}$$

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complement

$$\begin{array}{ccc} S & \xrightarrow{\ h\ } & T \\ \cup\!| & & \cup\!| \\ L & = & h^{-1}(U) \end{array} \qquad \begin{array}{ccc} S & \xrightarrow{\ h\ } & T \\ \cup\!| & & \cup\!| \\ S\setminus L & = & h^{-1}(T\setminus U) \end{array}$$

Properties of Recognisable Sets

Recognisable sets are closed under inverse homomorphisms

$$\begin{array}{ccc} S & \xrightarrow{\quad h \quad} & T \\ \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\ L & = & h^{-1}(U) \end{array}$$

Properties of Recognisable Sets

Recognisable sets are closed under inverse homomorphisms

$$S' \xrightarrow{\;g\;} S \xrightarrow{\;h\;} T$$

$$\begin{array}{ccccc} S' & \xrightarrow{g} & S & \xrightarrow{h} & T \\ \cup\!\!\shortmid & & \cup\!\!\shortmid & & \cup\!\!\shortmid \\ g^{-1}(L) = (h \circ g)^{-1}(U) & & L & = & h^{-1}(U) \end{array}$$

Logic in General

$\mathcal{R} = r_1, r_2, \ldots$ signature of relation symbols

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$\forall x$, $\exists x$	$\phi \wedge \psi$, $\phi \vee \psi$, $\neg \psi$	$r(x_1,\ldots,x_n)$	$x=y$
quantification over individuals	boolean operations	relations from the signature	equality

First Order Logic (FO)

Logic in General

$\mathcal{R} = r_1, r_2, \ldots$ signature of relation symbols

$\forall x, \exists x$ quantification over individuals

$\phi \wedge \psi, \phi \vee \psi, \neg \psi$ boolean operations

$r(x_1, \ldots, x_n)$ relations from the signature

$x = y$ equality

First Order Logic (FO)

$\forall X, \exists X$ quantification over sets

$x \in X$ membership

Monadic Second Order Logic (MSO)

Logic in General

$\mathcal{R} = r_1, r_2, \ldots$ signature of relation symbols

$S = (\mathcal{U}, r^S_1, r^S_2, \ldots)$ relational structure

universe

interpretation of relation symbols

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$\mathcal{R} = r_1, r_2, \ldots$ signature of relation symbols

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interpretation of relation symbols

$\mathcal{S}, u_1, \ldots, u_n, \mathcal{U}_1, \ldots, \mathcal{U}_m \models \phi(x_1, \ldots, x_n, \mathcal{X}_1, \ldots \mathcal{X}_m)$ the formula ϕ is true in $\mathcal{S}$ when

- x_i is mapped to the element u_i
- $\mathcal{X}_j$ is mapped to the set $\mathcal{U}_j$

Logic for Words (the precedence model)

$\mathcal{L} = \{a, b, c, \ldots\}$

finite alphabet

$x \leq y$

position x is before position y

$a(x)$

position x has letter a

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$\mathcal{L}(\phi) = \{ w \in \mathcal{E} \mid w \models \phi \}$ language of a sentence

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$\mathcal{L} \subseteq \mathcal{E}^*$ is FO (MSO) definable iff $\mathcal{L} = \mathcal{L}(\phi)$ for ϕ a FO (MSO) sentence

Some FO-definable Languages

$$x{+}1{=}y \stackrel{\text{def}}{=} \underbrace{x \leqslant y \wedge \neg x{=}y}_{x<y} \wedge \forall z \,.\, x \leqslant z \wedge z \leqslant y \Rightarrow x{=}z \vee z{=}y$$

Some FO-definable Languages

$$x{+}1{=}y \stackrel{\text{def}}{=} \underbrace{x{\leqslant}y \land \neg x{=}y}_{x<y} \land \forall z\ .\ x{\leqslant}z \land z{\leqslant}y \Rightarrow x{=}z \lor z{=}y$$

a^*bc^* $\quad \exists x\ .\ b(x) \land \forall y\ .\ y<x \Rightarrow a(x) \land \forall y\ .\ x<y \Rightarrow c(x)$

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$first(x) \stackrel{\text{def}}{=} \forall y \,.\, x \leq y \qquad last(x) \stackrel{\text{def}}{=} \forall y \,.\, y \leq x$

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$(ab)^+$ $\quad \exists x \,.\, first(x) \land a(x) \land \forall y \,.\, a(y) \Rightarrow \neg last(y) \land b(y+1) \land \forall y \,.\, b(y) \land \neg last(y) \Rightarrow a(y+1)$

Some MSO-definable Languages

$$\mathcal{X}\subseteq\mathcal{Y} \stackrel{\text{def}}{=} \forall x \,.\, x\in\mathcal{X} \Rightarrow x\in\mathcal{Y}$$

$$\mathcal{X}\subseteq a \stackrel{\text{def}}{=} \forall x \,.\, x\in\mathcal{X} \Rightarrow a(x)$$

$$sing(\mathcal{X}) \stackrel{\text{def}}{=} \mathcal{X}\neq\emptyset \wedge \forall\mathcal{Y} \,.\, \mathcal{Y}\subseteq\mathcal{X} \Rightarrow \mathcal{Y}=\emptyset \vee \mathcal{Y}=\mathcal{X}$$

$$\mathcal{X}\leqslant\mathcal{Y} \stackrel{\text{def}}{=} \forall x\, \forall y \,.\, x\in\mathcal{X} \wedge y\in\mathcal{Y} \Rightarrow x\leqslant y$$

Some MSO-definable Languages

$(aa)^+$ all words of non-zero even length over the alphabet $\mathcal{L} = \{a\}$

$$\exists X \,.\, \forall x \,.\, first(x) \Rightarrow x{\in}X \wedge last(x) \Rightarrow x{\notin}X \wedge \forall y \,.\, x{=}y{+}1 \Rightarrow (x{\in}X \Leftrightarrow y{\notin}X)$$

all words having a on even positions over the alphabet $\mathcal{L} = \{a, b\}$

$$\exists X \,.\, \forall x \,.\, first(x) \Rightarrow x{\in}X \wedge \forall y \forall z \,.\, y{\in}X \wedge z{=}y{+}1 \Rightarrow z{\notin}X \wedge X{\subseteq}a$$

As we shall see later, these languages are not FO-definable ...

Logic for Words (the successor model)

$\mathcal{L} = \{a, b, c, ...\}$ finite alphabet

$x+1=y$ — position y is the successor of x

$a(x)$ — position x has letter a

Makes a difference for FO-definability but not for MSO-definability

$$x \leq y \stackrel{\text{def}}{=} \forall X \,.\, (x \in X \wedge \underbrace{(\forall y \forall z \,.\, y \in X \wedge y+1=z \Rightarrow z \in X)}_{\text{closed under successors}}) \Rightarrow y \in X$$

closed under successors

MSO-definability = recognisability for words

Theorem [Trakhtenbrot-Büchi-Elgot]

Let $\mathcal{E}$ be a finite alphabet and L ⊆ $\mathcal{E}^*$ be a language.

L is recognisable ⇔ L is MSO-definable.

"⇒" L is recognisable ⇒ L is the language of a finite automaton A with n states

$\exists X_1 \ldots \exists X_n$. *the sets* $X_1 \ldots X_n$ *describe an accepting run of* A

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"⇐" $X \subseteq Y$ $X \subseteq a$ $sing(X)$ $X \leqslant Y$

each MSO formula can be written as $\phi(X_1 \ldots X_n)$

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each MSO formula can be written as $\phi(X_1 \dots X_n)$

the language of $\phi(X_1 \dots X_n)$ is the set of words w
over the alphabet $\mathcal{L} \times \{0,1\}^n$: $w, U_1, \dots, U_n \models \phi(X_1 \dots X_n)$
where $i \in U_j \Leftrightarrow$ the j-th bit on position i is 1

X_1	0	1	⋯	0
X_2	1	0	⋯	1
⋮	⋮	⋮		⋮
X_n	0	0	⋯	1
$\mathcal{L}$	a_1	a_2	...	a_m

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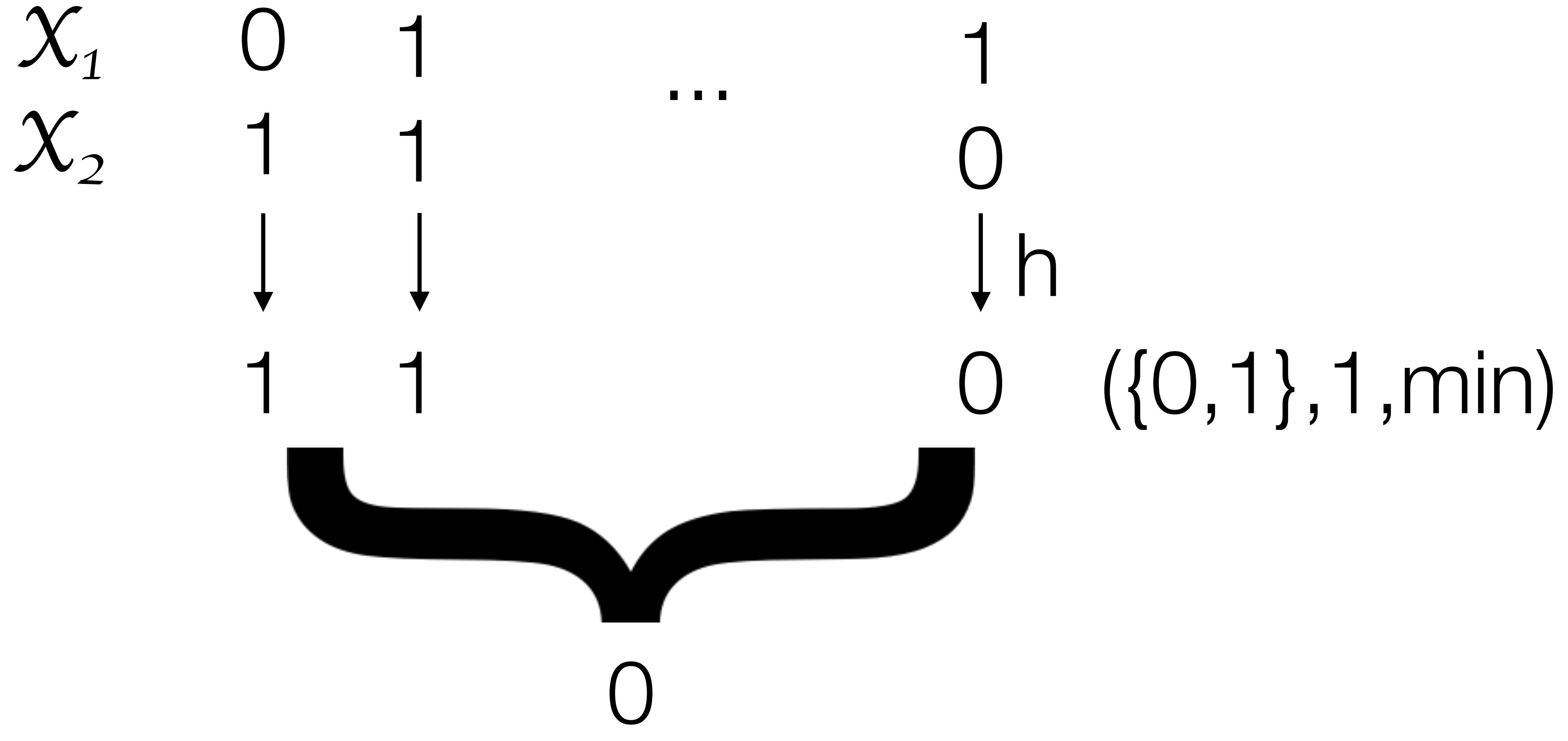

$X_1 \subseteq a$

similar construction

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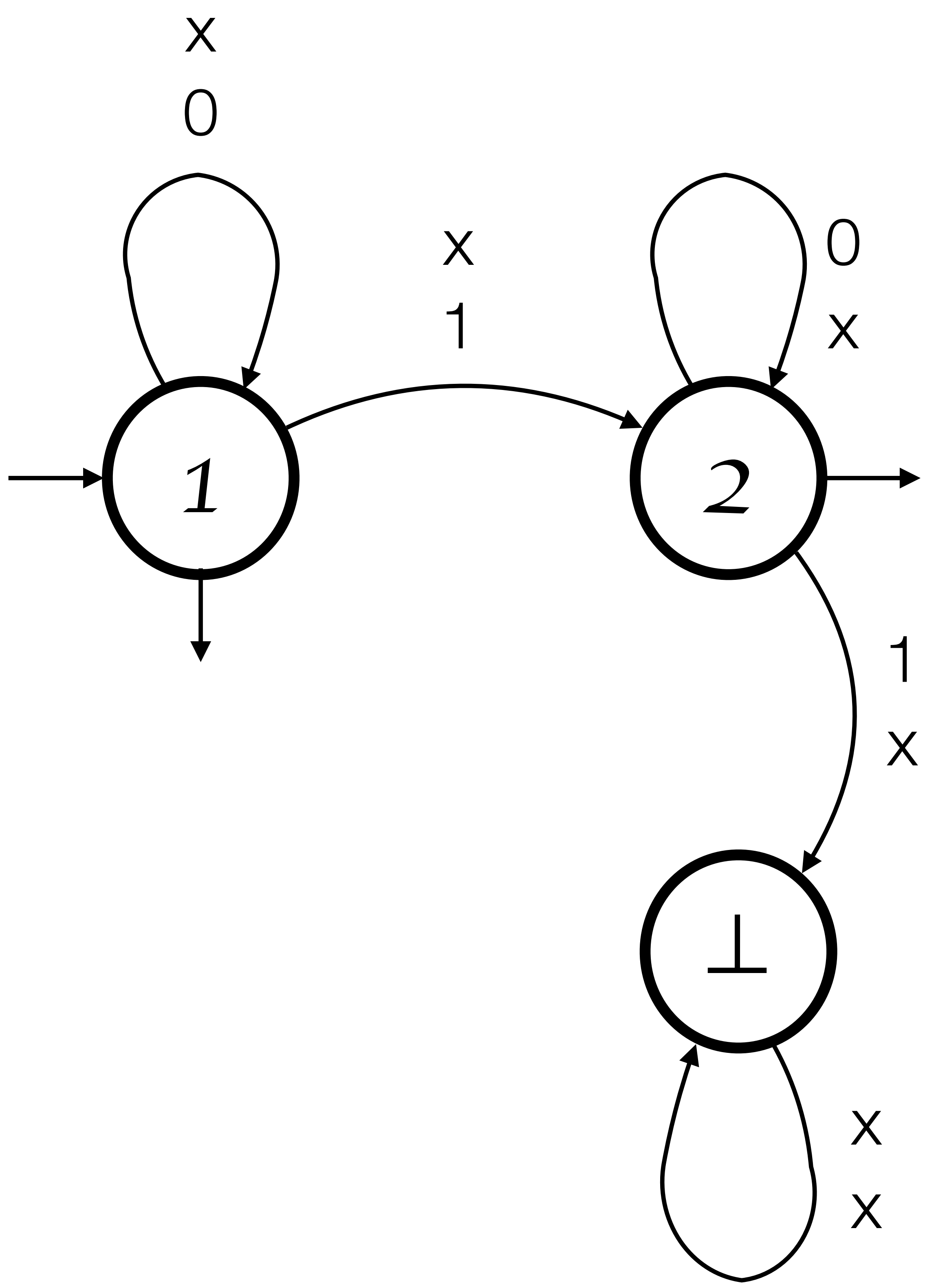

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"⇐" $\phi \wedge \psi$, $\phi \vee \psi$, $\neg \psi$ closure of recognizable sets under boolean operations

$$(\mathcal{L} \times \{0,1\}^{n-1})^* \xleftarrow{\pi} (\mathcal{L} \times \{0,1\}^{n})^* \xrightarrow{h} M$$

$$\phi(X_1 \dots X_n) \subseteq (\mathcal{L} \times \{0,1\}^{n})^*, \quad N \subseteq M$$

$$\phi(X_1 \dots X_n) = h^{-1}(N)$$

$$(\mathcal{L} \times \{0,1\}^{n-1})^* \xrightarrow{H(w)=\{h(v) \mid \pi(v)=w\}} 2^M$$

$$\exists X_n . \phi(X_1 \dots X_n) \subseteq (\mathcal{L} \times \{0,1\}^{n-1})^*, \quad \{P \subseteq M \mid P \cap N \neq \varnothing\} \subseteq 2^M$$

$$\exists X_n . \phi(X_1 \dots X_n) = H^{-1}(\{P \subseteq M \mid P \cap N \neq \varnothing\})$$