I. Recognisability by Semigroups and Monoids

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Semigroups

$$a, b \in S \Rightarrow a \cdot b \in S$$

$(S, \cdot)$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \qquad abc \overset{\text{def}}{=} a \cdot b \cdot c$$

Example 1 $(\{0,1,...,n-1\}, +_n)$ $x +_n y \overset{\text{def}}{=} x+y \bmod n$

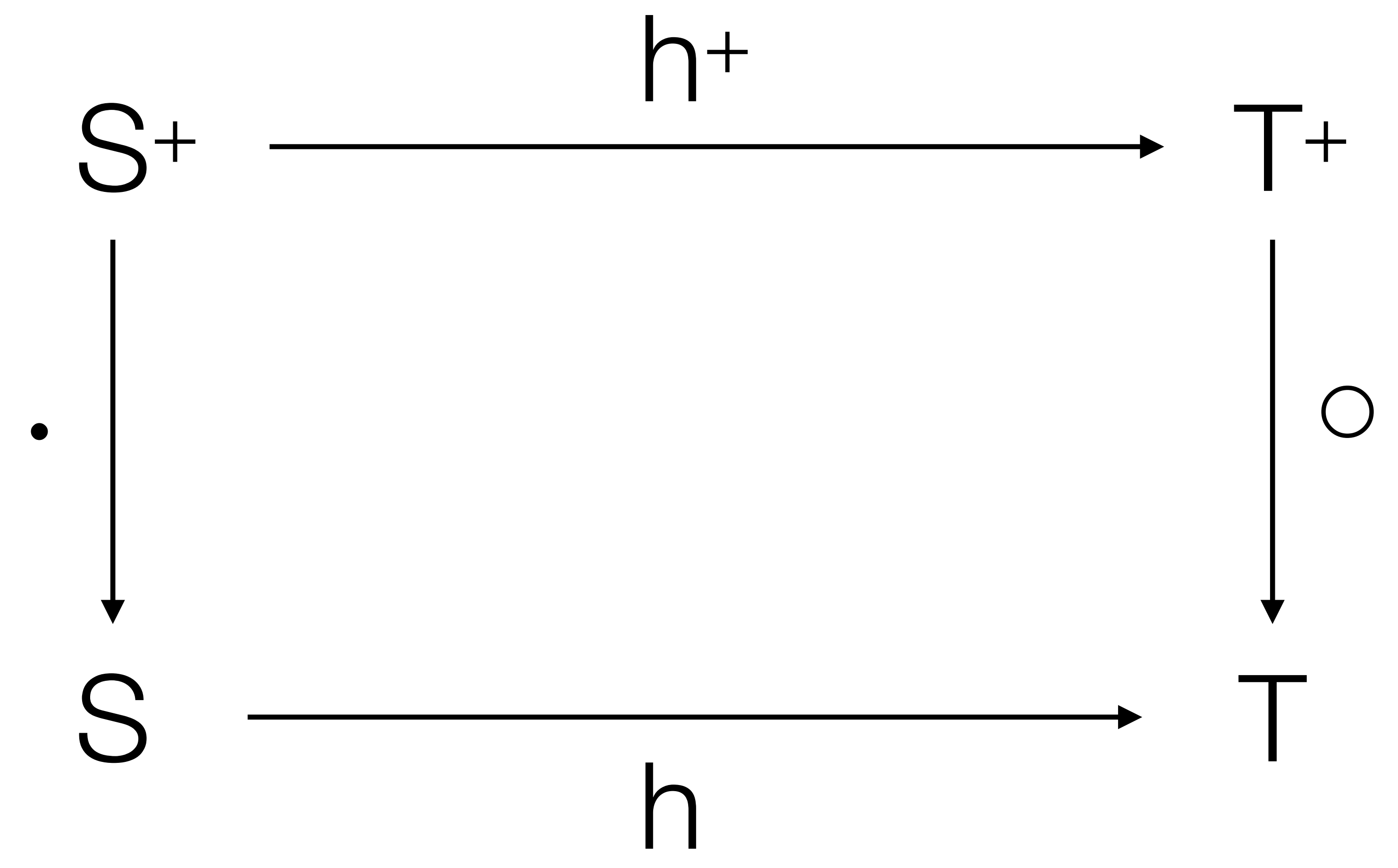
Example 2 $(2^{A \times A}, \circ)$ $P \circ Q \overset{\text{def}}{=} \{(x,y) \mid \exists z \in A . (x,z) \in P \text{ and } (z,y) \in Q\}$

Example 3 $(\Sigma^+, \cdot)$ $u \cdot v \overset{\text{def}}{=} uv$

Semigroup Homomorphism

$h : (S, \cdot) \rightarrow (T, \circ)$ $h(a \cdot b) = h(a) \circ h(b)$

$$h^+(a_1, a_2, ..., a_n) \stackrel{\text{def}}{=} h(a_1), h(a_2), ..., h(a_n)$$



Monoids

$$a, b \in S \Rightarrow a \cdot b \in S$$

$(S, 1, \cdot\,)$ $\quad$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ $\quad$ $abc \overset{def}{=} a \cdot b \cdot c$

$$1 \cdot a = a \cdot 1 = a$$

Example 1 $\quad$ $(\{0,1,\ldots,n-1\}, 0, +_n)$ $\quad$ $x +_n y \overset{def}{=} x+y \bmod n$

Example 2 $\quad$ $(2^{A\times A}, Id_A, \circ)$ $\quad$ $P\circ Q \overset{def}{=} \{(x,y) \mid \exists z\in A \,.\, (x,z)\in P \text{ and } (z,y)\in Q\}$

$$Id_A \overset{def}{=} \{(x,x) \mid x\in A\}$$

Example 3 $\quad$ $(\Sigma^*, \varepsilon, \cdot\,)$ $\quad$ $u \cdot v \overset{def}{=} uv$

Monoid Homomorphism

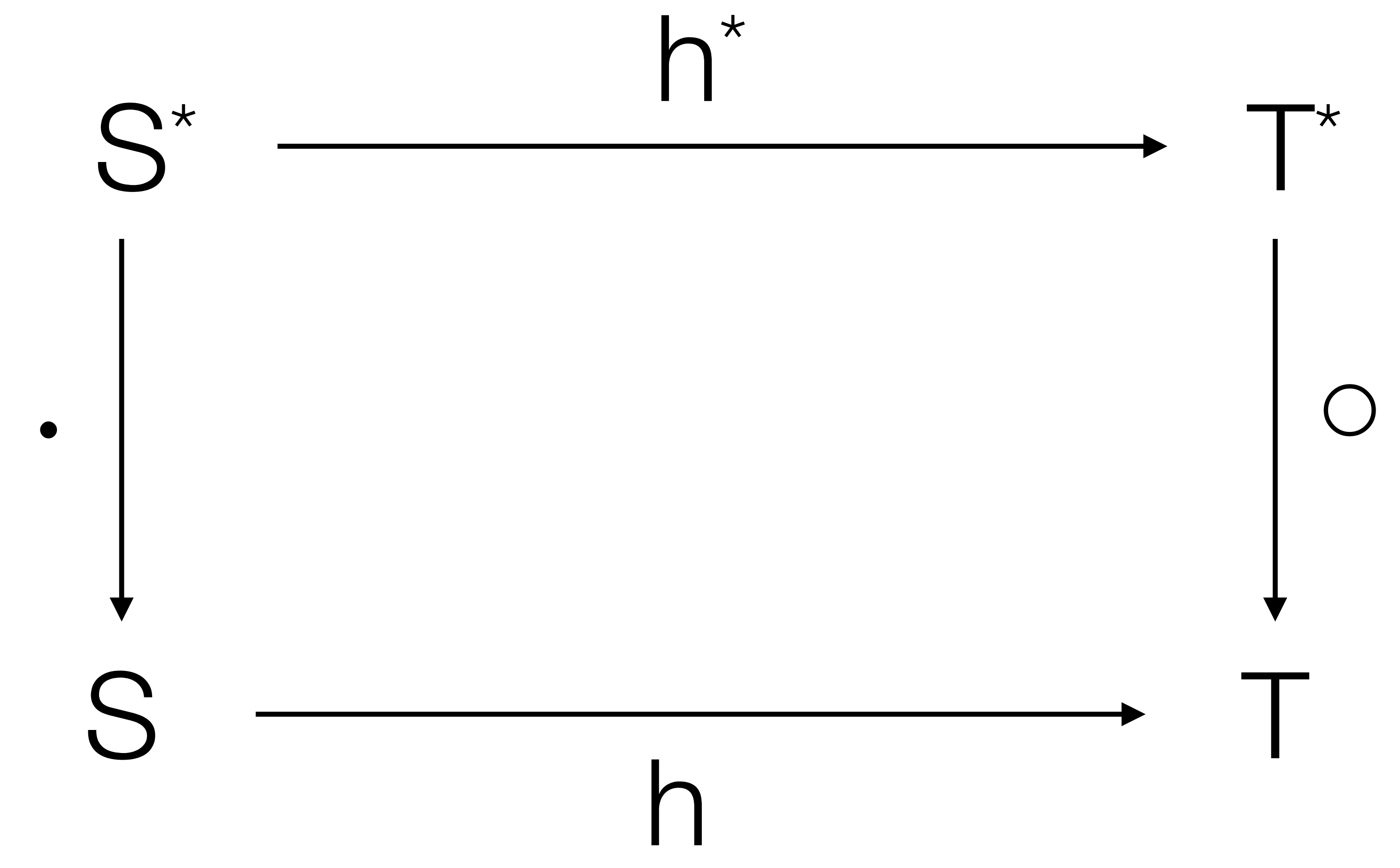
$h : (S, 1_S, \cdot\,) \rightarrow (T, 1_T, \circ)$

$h(a \cdot b) = h(a) \circ h(b)$

$h(1_S) = 1_T$

$h^*(a_1, a_2, \ldots, a_n) \stackrel{\text{def}}{=} h(a_1), h(a_2), \ldots, h(a_n)$

$h^*(\varepsilon) \stackrel{\text{def}}{=} \varepsilon$



Recognisability

$(S, \cdot)$ semigroup (same definition for S monoid)

$$S$$

$$\cup\!\!|$$

$$L$$

Recognisability

(S, ·) semigroup (same definition for S monoid)

$$S \xrightarrow{\quad h \quad} T$$ finite semigroup

$\cup$I

L

Recognisability

(S, ·) semigroup (same definition for S monoid)

$$\begin{array}{ccl} S & \xrightarrow{\quad h \quad} & T \text{ finite semigroup} \\ \rotatebox{-90}{$\supseteq$} & & \rotatebox{-90}{$\supseteq$} \\ L & & U \text{ accepting subset} \end{array}$$

Recognisability

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$$\begin{array}{ccc} S & \xrightarrow{\quad h \quad} & T \text{ finite semigroup} \\ \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\ L & = & h^{-1}(U) \text{ accepting subset} \end{array}$$

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$$S \xrightarrow{\quad h \quad} T$$ finite semigroup

$$\rotatebox{90}{$\subseteq$} \qquad\qquad\qquad \rotatebox{90}{$\subseteq$}$$

$$L \quad = \quad h^{-1}(U)$$ accepting subset

Example 1 The set of odd (even) numbers is recognisable in $(\mathbb{N}, +)$

Recognisable Word Languages

Theorem $L \subseteq \Sigma^*$ is recognisable in the monoid $(\Sigma^*, \varepsilon, \cdot)$

$\Leftrightarrow$

L is the language of a nondeterministic finite automaton $A = (Q, I, F, \rightarrow)$

"⇒"

$$\begin{array}{ccc} \Sigma^* & \xrightarrow{\quad h \quad} & (Q, 1_Q, \circ) \text{ finite monoid} \\ \cup\!\!\!\mid & & \cup\!\!\!\mid \\ L & = & h^{-1}(F) \end{array}$$

$A = (Q, \{1_Q\}, F, \rightarrow)$ $\qquad q \xrightarrow{a} q \circ h(a)$

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$$\Sigma^* \xrightarrow{h(w)=\{ (q,q') \mid q \overset{w}{\rightarrow} q'\}} (2^{Q \times Q}, Id_Q, \circ)$$

$$\text{\rotatebox{90}{$\subseteq$}} \qquad\qquad\qquad \text{\rotatebox{90}{$\subseteq$}}$$

$$L \quad = \quad h^{-1}(\{R \mid \exists i \in I \ \exists f \in F \ . \ (i,f) \in R\})$$

Syntactic Monoids

$(S, 1_S, \cdot)$ monoid $\quad \sim \subseteq S \times S \quad a \sim b \Rightarrow sat \sim sbt$, for all $s, t \in S$ congruence

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Theorem [Myhill-Nerode]

$L \subseteq S$ is recognisable $\Leftrightarrow S_{/\sim L}$ is finite

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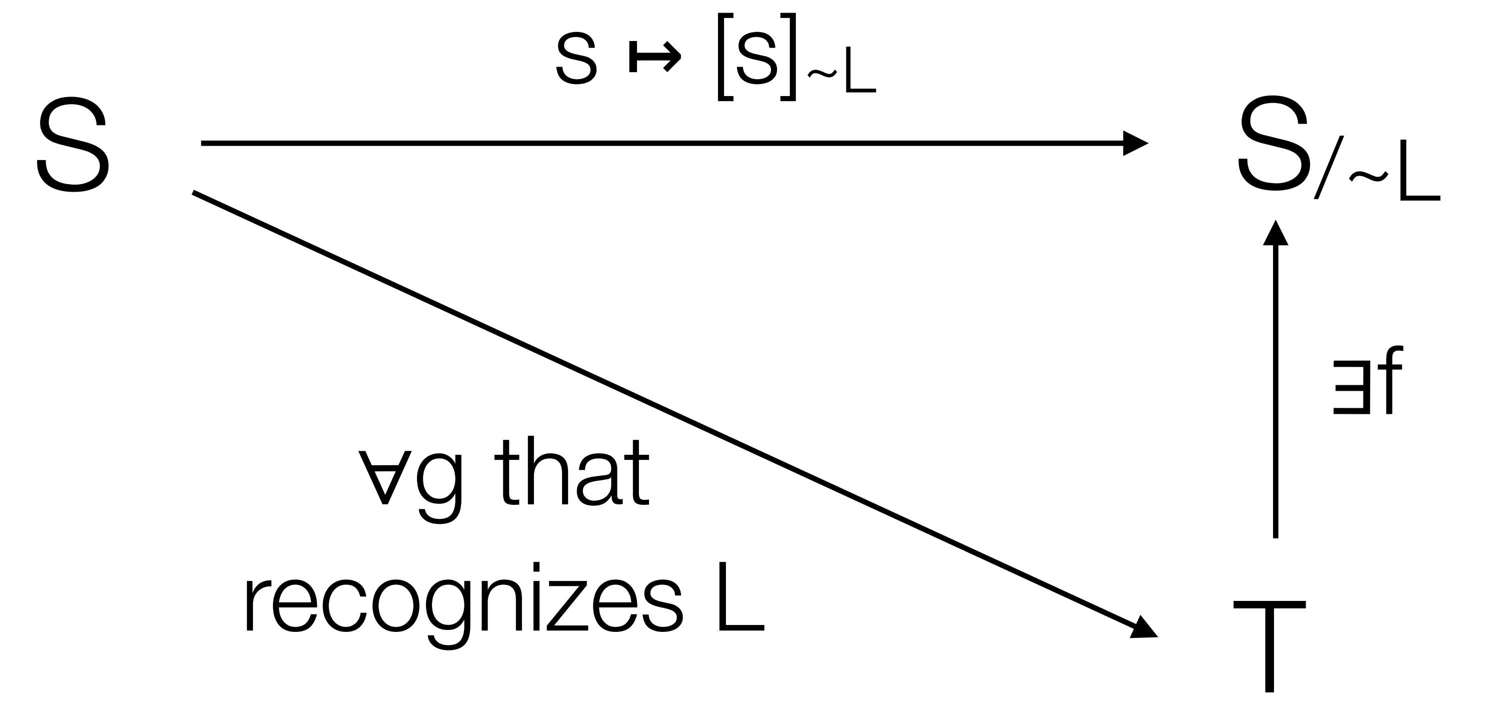
Theorem [Myhill-Nerode]

$L \subseteq S$ is recognisable $\Leftrightarrow$ $S_{/\sim L}$ is finite



all homomorphisms are surjective

Properties of Recognisable Sets

Recognisable sets are closed under boolean combinations

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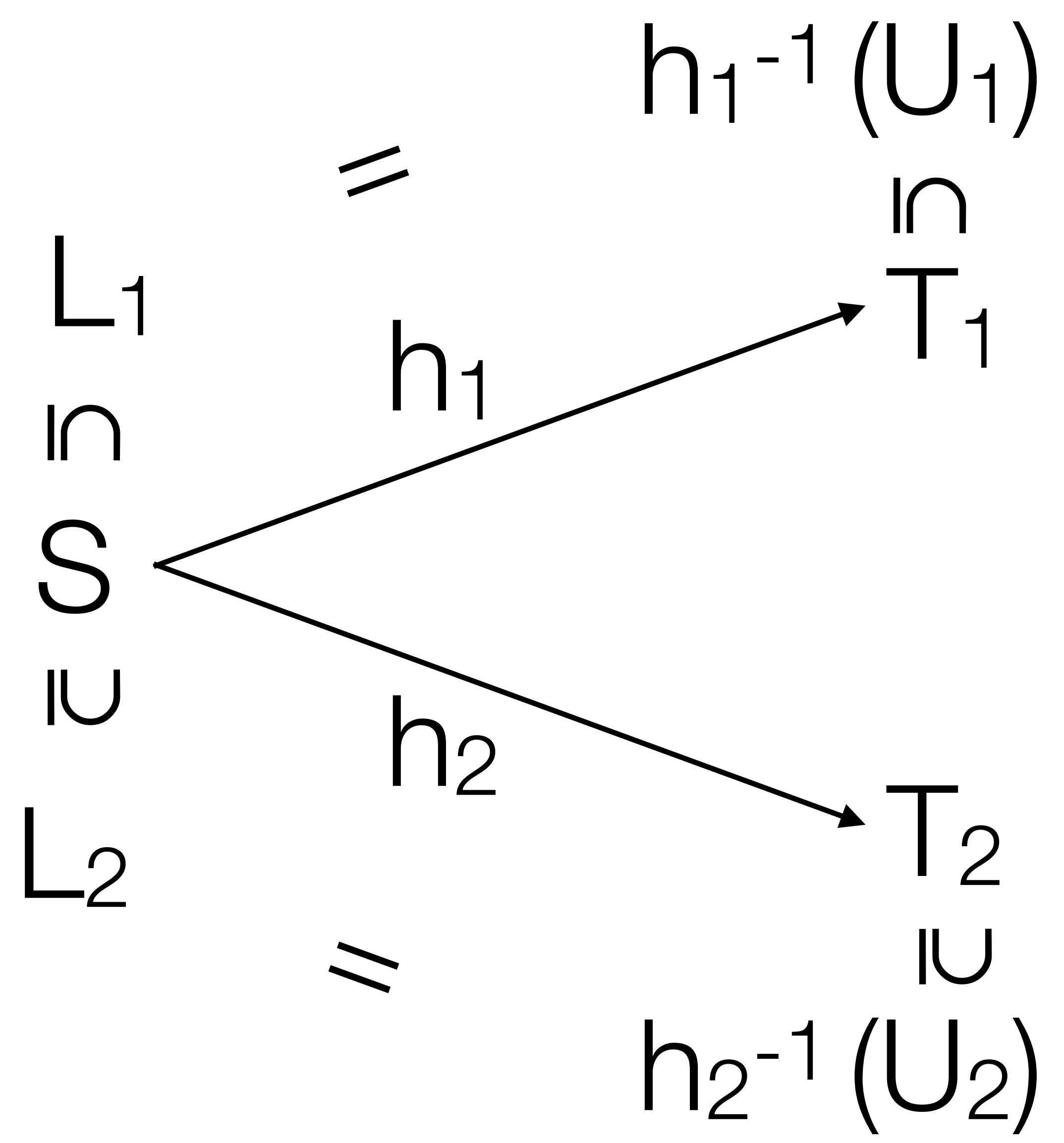
Recognisable sets are closed under boolean combinations

intersection



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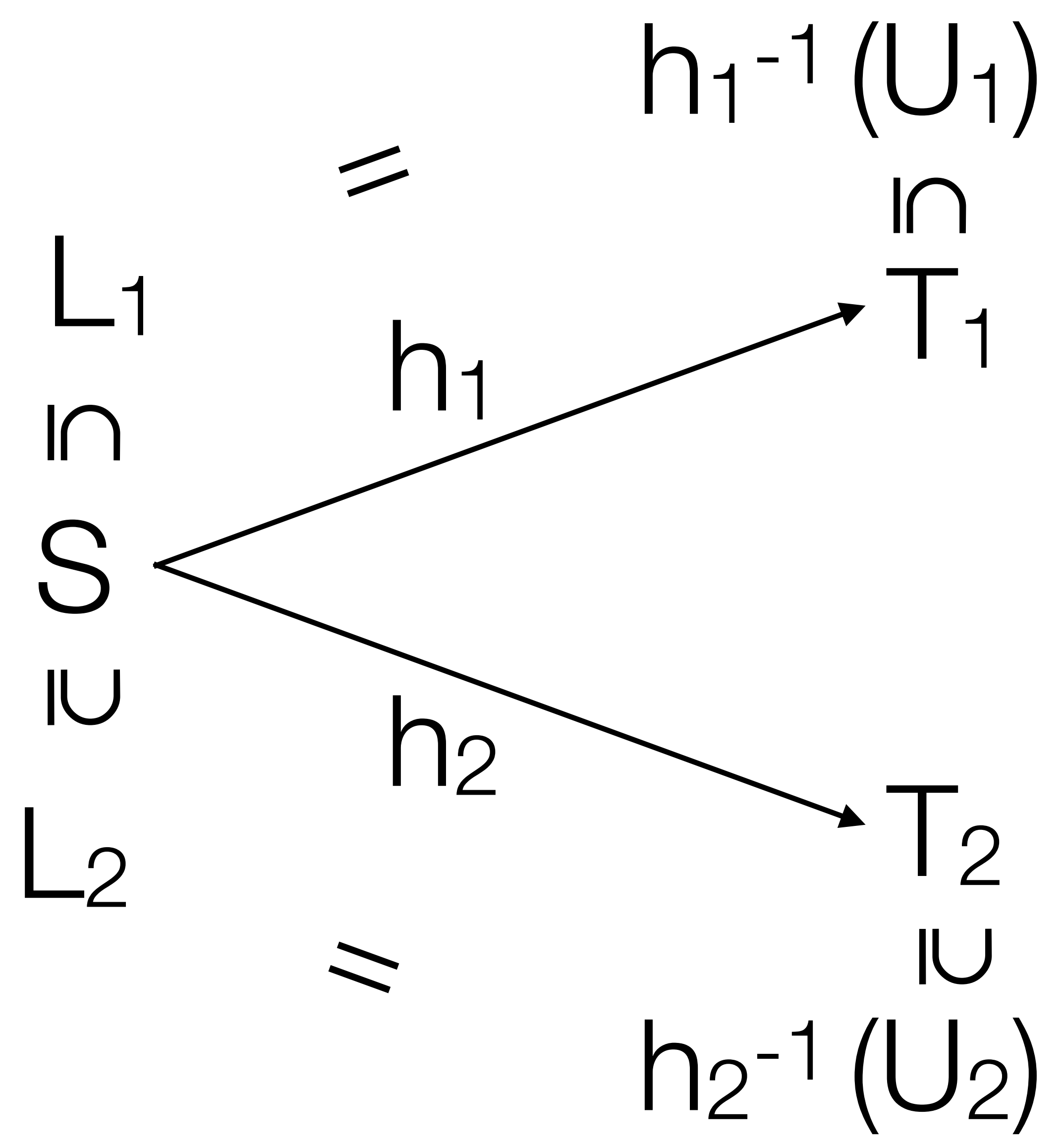
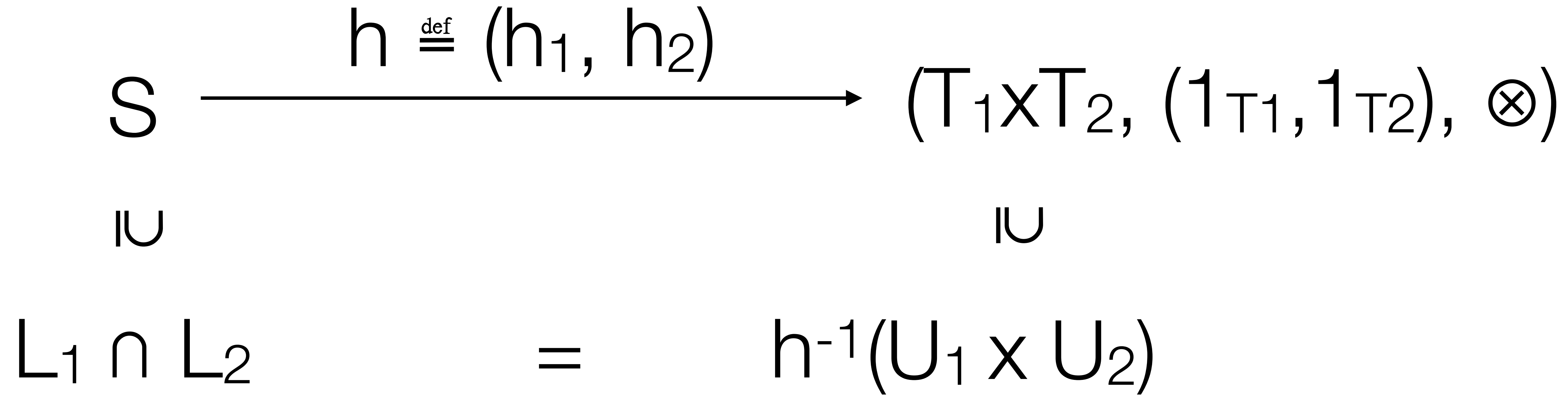


$$(s_1,s_2) \otimes (t_1,t_2) \overset{\text{def}}{=} (s_1t_1,\ s_2t_2)$$



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$$\begin{array}{ccc} S & \xrightarrow{\quad h \quad} & T \\ \cup\!| & & \cup\!| \\ L & = & h^{-1}(U) \end{array}$$

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complement

$$\begin{array}{ccc} S & \xrightarrow{\quad h \quad} & T \\ \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\ L & = & h^{-1}(U) \end{array} \qquad\qquad \begin{array}{ccc} S & \xrightarrow{\quad h \quad} & T \\ \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\ S \setminus L & = & h^{-1}(T \setminus U) \end{array}$$

Properties of Recognisable Sets

Recognisable sets are closed under inverse homomorphisms

$$
\begin{array}{ccc}
S & \xrightarrow{\quad h \quad} & T \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
L & = & h^{-1}(U)
\end{array}
$$

Properties of Recognisable Sets

Recognisable sets are closed under inverse homomorphisms

$$S' \xrightarrow{g} S \xrightarrow{h} T$$

$$\cup\!\!| \qquad \cup\!\!| \qquad \cup\!\!|$$

$$g^{-1}(L) = (h \circ g)^{-1}(U) \qquad L \quad = \quad h^{-1}(U)$$

Logic in General

$\mathcal{R} = r_1, r_2, ...$ signature of relation symbols

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$\forall x, \exists x$	$\varphi \wedge \psi,\ \varphi \vee \psi,\ \neg\psi$	$r(x_1,\ldots,x_n)$	$x=y$
quantification over individuals	boolean operations	relations from the signature	equality

First Order Logic (FO)

Logic in General

$\mathcal{R} = r_1, r_2, \ldots$ signature of relation symbols

$\forall x, \exists x$ quantification over individuals

$\varphi \wedge \psi, \varphi \vee \psi, \neg\psi$ boolean operations

$r(x_1, \ldots, x_n)$ relations from the signature

$x = y$ equality

First Order Logic (FO)

$\forall \mathcal{X}, \exists \mathcal{X}$ quantification over sets

$x \in \mathcal{X}$ membership

Monadic Second Order Logic (MSO)

Logic in General

$\mathcal{R} = r_1, r_2, \ldots$ signature of relation symbols

$S = (\mathcal{U}, r^S_1, r^S_2, \ldots)$ relational structure

universe

interpretation of relation symbols

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$S, u_1, \ldots, u_n, \mathcal{U}_1, \ldots, \mathcal{U}_m \models \varphi\,(x_1, \ldots, x_n, \mathcal{X}_1, \ldots \mathcal{X}_m)$ the formula φ is true in S when

- x_i is mapped to the element u_i
- $\mathcal{X}_j$ is mapped to the set $\mathcal{U}_j$

Logic for Words

$\Sigma = \{a,b,c,...\}$

finite alphabet

$x \leq y$

position x is before position y

$a(x)$

position x has letter a

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$\mathcal{L} \subseteq \Sigma^*$ is FO (MSO) definable iff $\mathcal{L} = \mathcal{L}(\varphi)$ for φ a FO (MSO) sentence

Some FO-definable Languages

$$x{+}1{=}y \stackrel{\text{def}}{=} \underbrace{x{\leqslant}y \land \neg x{=}y}_{x<y} \land \forall z\ .\ x{\leqslant}z \land z{\leqslant}y \Rightarrow x{=}z \lor z{=}y$$

Some FO-definable Languages

$$x{+}1{=}y \stackrel{\text{def}}{=} \underbrace{x{\leqslant}y \wedge \neg x{=}y}_{x<y} \wedge \forall z\ .\ x{\leqslant}z \wedge z{\leqslant}y \Rightarrow x{=}z \vee z{=}y$$

a^*bc^* $\quad \exists x\ .\ b(x) \wedge \forall y\ .\ y{<}x \Rightarrow a(x) \wedge \forall y\ .\ x{<}y \Rightarrow c(x)$

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$first(x) \stackrel{\text{def}}{=} \forall y \,.\, x \leq y \qquad last(x) \stackrel{\text{def}}{=} \forall y \,.\, y \leq x$

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$$first(x) \stackrel{\text{def}}{=} \forall y \,.\, x \leq y \qquad\qquad last(x) \stackrel{\text{def}}{=} \forall y \,.\, y \leq x$$

$$(ab)^+ \qquad \exists x \,.\, first(x) \wedge a(x) \wedge \forall y \,.\, a(y) \Rightarrow \neg last(y) \wedge b(y{+}1) \wedge \forall y \,.\, b(y) \wedge \neg last(y) \Rightarrow a(y{+}1)$$

Some MSO-definable Languages

$(aa)^+$ all words of non-zero even length over the alphabet $\Sigma = \{a\}$

$$\exists X \,.\, \forall x \,.\, first(x) \Rightarrow x \in X \wedge last(x) \Rightarrow x \notin X \wedge \forall y \,.\, x=y+1 \Rightarrow (x \in X \Leftrightarrow y \notin X)$$

all words having a on even positions over the alphabet $\Sigma = \{a, b\}$

$$\exists X \,.\, \forall x \,.\, first(x) \Rightarrow x \in X \wedge \forall y \,.\, x \in X \wedge x=y+1 \Rightarrow y \notin X \wedge \forall x \,.\, X(x) \Rightarrow a(x)$$

As we shall see later, these languages are not FO-definable ...

Logic for Words (the successor model)

$\Sigma = \{a, b, c, ...\}$ finite alphabet

$x{+}1{=}y$ $\{a(x) \mid a{\in}\Sigma\}$ signature

position y is the successor of x

position x has letter a

Makes a difference for FO-definability but not for MSO-definability

$$x{\leq}y \stackrel{\text{def}}{=} \forall X\ .\ (x{\in}X \wedge (\forall y \forall z\ .\ y{\in}X \wedge \underbrace{z{\in}X \wedge y{+}1{=}z}_{\text{closed under successors}} \Rightarrow z{\in}X)) \Rightarrow y{\in}X$$

MSO-definability = recognisability for words

Theorem [Trakhtenbrot-Büchi-Elgot] Let Σ be a finite alphabet.

Each language $L \subseteq \Sigma^*$ is recognisable ⇔ L is MSO-definable

"⇒" L is recognisable ⇒ L is the language of a finite automaton A with n states

$\exists X_1 \ldots \exists X_n$. *the sets* $X_1 \ldots X_n$ *describe an accepting run of* A

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"⇐" $\quad \mathcal{X} \subseteq \mathcal{Y} \quad \mathcal{X} \leqslant \mathcal{Y} \quad \mathcal{X} \subseteq a \quad \mathcal{X} \neq \emptyset \wedge \forall \mathcal{Y} .\ \mathcal{Y} \subseteq \mathcal{X} \Rightarrow \mathcal{Y} = \emptyset \vee \mathcal{Y} = \mathcal{X}$

$x \leqslant y$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ $\quad$ $a(x)$ for all $x \in \mathcal{X}$ $\quad$ $\mathcal{X}$ is a singleton

each MSO formula can be written as $\varphi\,(\mathcal{X}_1 \ldots \mathcal{X}_n)$

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each MSO formula can be written as $\varphi(\mathcal{X}_1 \ldots \mathcal{X}_n)$

the language of $\varphi(\mathcal{X}_1 \ldots \mathcal{X}_n)$ is the set of words w

over the alphabet $\Sigma \times \{0,1\}^n$: $w, \mathcal{U}_1, \ldots, \mathcal{U}_n \models \varphi(\mathcal{X}_1 \ldots \mathcal{X}_n)$

where $i \in \mathcal{U}_j \Leftrightarrow$ the j-th bit on position i is 1

$\mathcal{X}_1$	0	1	⋯	0
$\mathcal{X}_2$	1	0	⋯	1
⋮	⋮	⋮		⋮
$\mathcal{X}_n$	0	0	⋯	1
Σ	a_1	a_2	...	a_m

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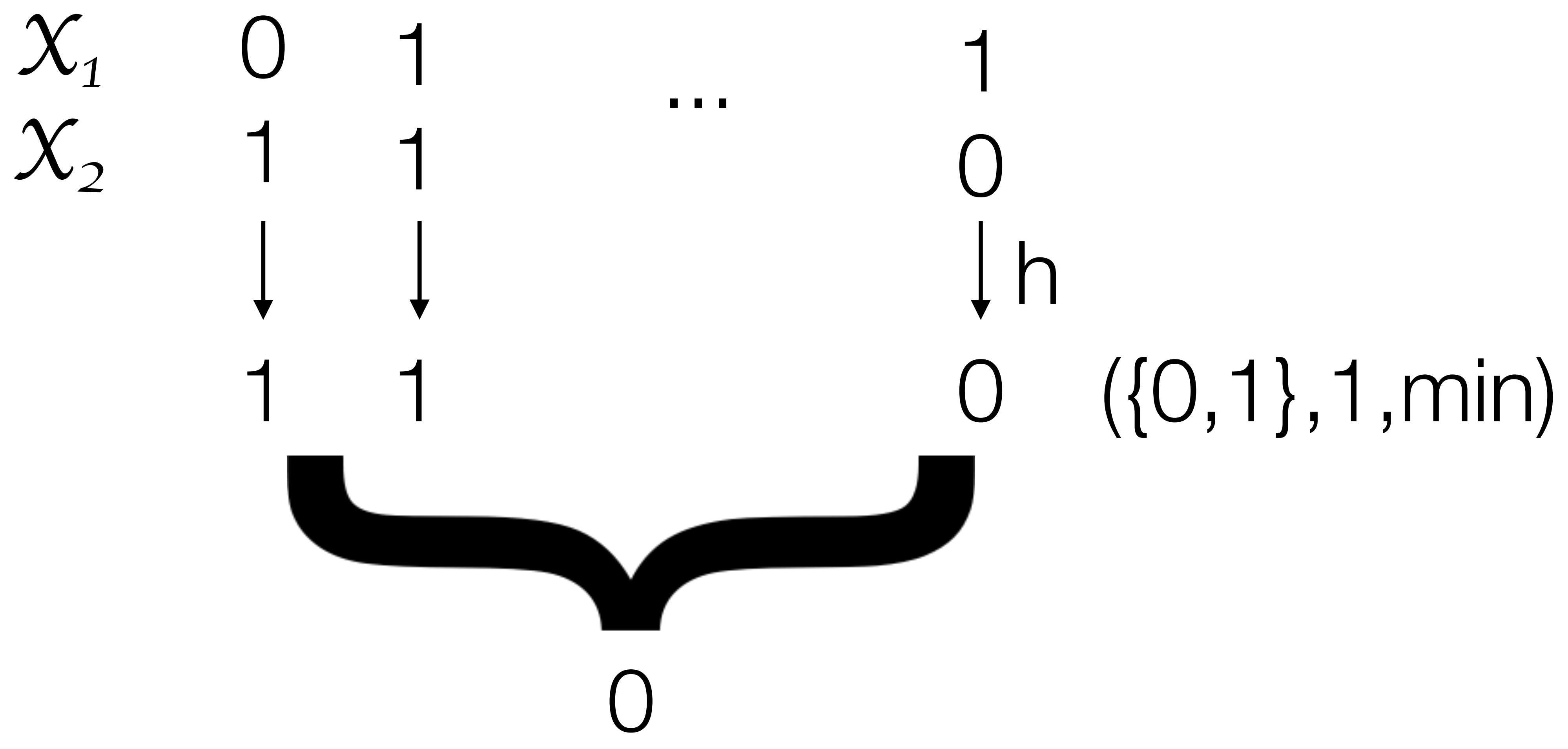
"⇐" $\mathcal{X}_1 \subseteq \mathcal{X}_2$



$\mathcal{X}_1 \subseteq a$

similar construction

MSO-definability = recognisability for words

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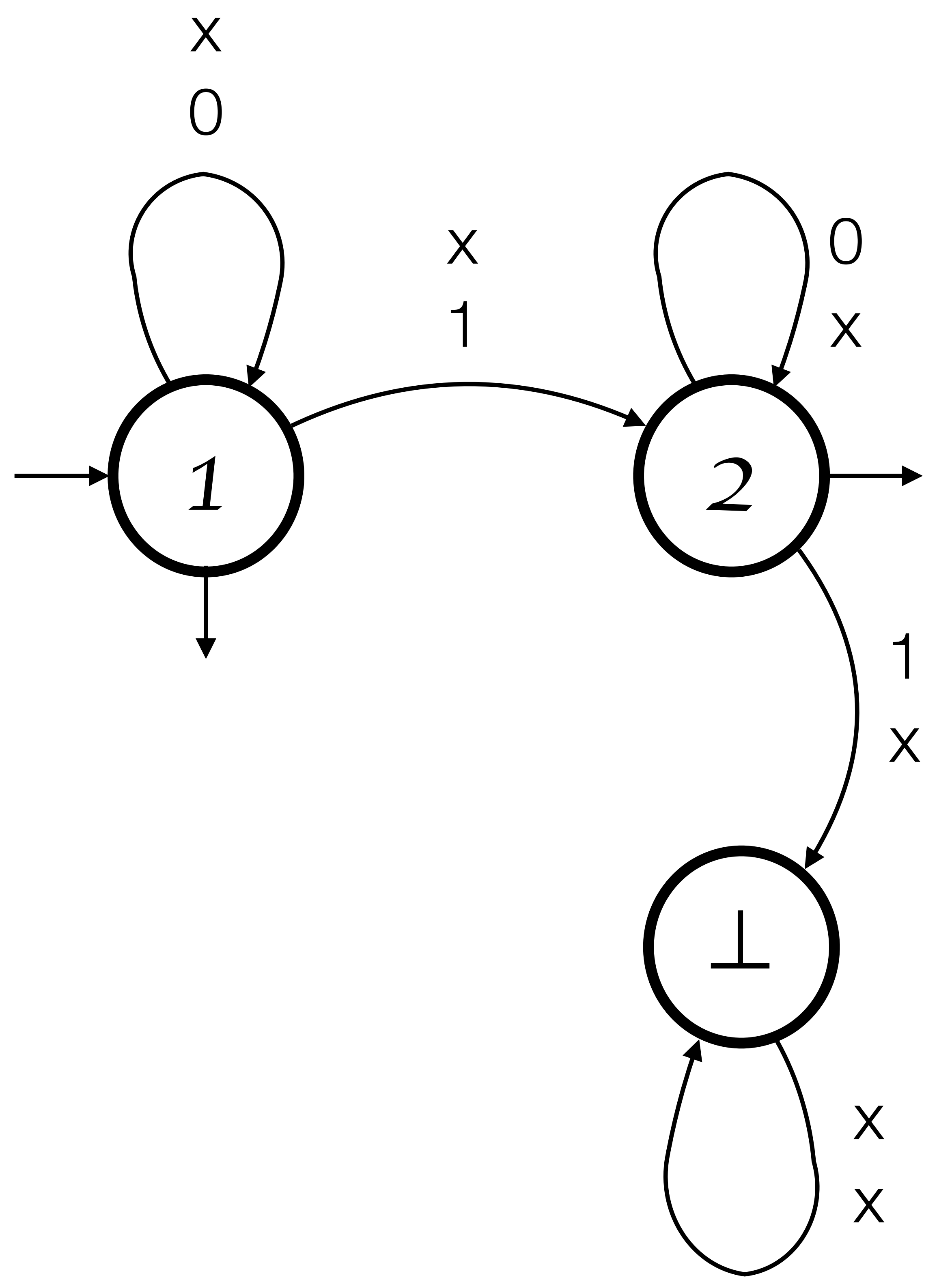

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"⇐" $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\psi$ closure of recognizable sets under boolean operations

$$(\Sigma \times \{0,1\}^{n-1})^* \xleftarrow{\pi} (\Sigma \times \{0,1\}^{n})^* \xrightarrow{h} M$$

$$\cup\!\!| \qquad \cup\!\!|$$

$$\varphi(X_1 \dots X_n) = h^{-1}(N)$$

$$(\Sigma \times \{0,1\}^{n-1})^* \xrightarrow{H(w)=\{h(v) \mid \pi(v)=w\}} 2^M$$

$$\cup\!\!| \qquad \cup\!\!|$$

$$\exists X_n . \varphi(X_1 \dots X_n) = H^{-1}(\{P \subseteq M \mid P \cap N = \varnothing\})$$