Introduction to Logic and Automata Theory

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Ensuring Correctness of Hw/Sw Systems

- Uses logic to specify correctness properties, e.g.:
  - *the program never crashes*
  - *the program always terminates*
  - *every request to the server is eventually answered*
  - *the output of the tree balancing function is a tree, provided the input is also a tree ...*
- Given a logical specification, we can do either:
  - VERIFICATION: prove that a given system satisfies the specification
  - SYNTHESIS: build a system that satisfies the specification

Approaches to Verification

- THEOREM PROVING: reduce the verification problem to the satisfiability of a logical formula (entailment) and invoke an off-the-shelf theorem prover to solve the latter
  - Floyd-Hoare checking of pre-, post-conditions and invariants
  - Certification and Proof-Carrying Code

- MODEL CHECKING: enumerate the states of the system and check that the transition system satisfies the property
  - explicit-state model checking (SPIN)
  - symbolic model checking (SMV)

- COMBINED METHODS:
  - static analysis (ASTREE)
  - predicate abstraction (SLAM, BLAST)

Approaches to Synthesis

- TREE AUTOMATA:
  - starting point: logical specification
  - build word automaton from logic formula
  - transform into tree automaton
  - decide emptiness and build system from witness tree
- CONTROL and GAME THEORY:
  - starting point: incomplete/uncontrolled system with two types of freedom (system/environment choice) and an objective
  - the uncontrolled system is given as a game
  - controller/strategy tell how to achieve objective

Logic and Automata Connection

Given an automaton A, we build a logical formula φ_A whose set of models is exactly the language of the automaton.

Given a logical formula φ, we build an automaton A_φ that recognizes the set of all structures (models) in which φ holds.

Assuming that A_φ belongs to a well-behaved class of automata, we can tackle the following problems:

- SATISFIABILITY: φ has a model if and only if A_φ is not empty
- MODEL CHECKING: a given structure is a model of φ if and only if it belongs to the language of A_φ

Overview: Word and Tree Logics

	First Order Logic	$\subset$	Monadic Second Order Logic
finite words	LTL, Star Free, Aperiodic Sets		Finite Automata
infinite words	LTL, Star Free, Aperiodic Sets		Büchi, Rabin Automata
finite trees	*		Tree Automata
infinite trees	*		Rabin Automata, Games

Overview: Integer Logics

Presburger Arithmetic $\quad \subset \quad \langle \mathbb{N}, +, V_p \rangle$

Semilinear Sets $\qquad\qquad$ p-automata

(provided as additional material)

Preliminaries

Words

An *alphabet* is a finite non-empty set of symbols $\Sigma = \{a, b, c, \ldots\}$.

A *word* of length n over Σ is a sequence $w = a_0 a_1 \ldots a_{n-1}$, where $a_i \in \Sigma$, for all $0 \leq i < n$. An *infinite word* is an infinite sequence of elements of Σ.

Equivalently, a word is a function $w : \{0, 1, \ldots, n-1\} \to \Sigma$. The *length* n of the word w is denoted by $|w|$. The *empty word* is denoted by ϵ, i.e. $|\epsilon| = 0$. An infinite word is a function $w : \mathbb{N} \to \Sigma$.

Σ^* (Σ^ω) is the set of all finite (infinite) words over Σ, and $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$. We denote $\Sigma^+ = \Sigma^* \setminus \{\epsilon\}$.

The *concatenation* of two words w and u is denoted as wu. Note that $w \in \Sigma^*$, whereas $u \in \Sigma^\infty$. The *prefix* u of w is defined as $u \leq w$ iff there exists $v \in \Sigma^*$ such that $uv = w$.

Trees

A *prefix-closed* set $S \subseteq \Sigma^*$ is such that for all $w \in S$ and $u \in \Sigma^*$, $u \leq w \;\Rightarrow\; u \in S$.

A *prefix-free* set $S \subseteq \Sigma^*$ is such that for all $u, v \in S$, $u \neq v \;\Rightarrow\; u \not\leq v$ and $v \not\leq u$.

A *tree* over Σ is a partial function $t : \mathbb{N}^* \mapsto \Sigma$ such that $dom(t)$ is a prefix-closed set.

The *children* of a tree node $w \in dom(t)$ are all nodes $wn \in dom(t)$, such that $n \in \mathbb{N}$. A tree t is said to be *finite-branching* iff for all $p \in dom(t)$, the number of children of p is finite. A tree t is said to be *finite* if $dom(t)$ is finite.

Trees (contd)

A *path* π is a set of nodes from $dom(t)$, such that:

1. the root belongs to the path i.e., $\epsilon \in \pi$,
2. for each node $p \in \pi$, exactly one of its children (if any) is on π.
3. for each node $pn \in \pi$, such that $n \in \mathbb{N}$, we have $p \in \pi$.

Lemma 1 (König) *A finitely branching tree is infinite if and only if it has an infinite path.*

Ranked Trees

A *ranked alphabet* $\langle \Sigma, \# \rangle$ is a set of symbols together with a function $\# : \Sigma \to \mathbb{N}$. For $f \in \Sigma$, the value $\#(f)$ is said to be the *arity* of f.

A *ranked tree* t over Σ is a partial function $t : \mathbb{N}^* \mapsto \Sigma$ that satisfies the following conditions:

- $dom(t)$ is a prefix-closed subset of $\mathbb{N}^*$, and
- for each $p \in dom(t)$, if $\#(t(p)) > 0$ then $\{i \mid pi \in dom(t)\} = \{1, \ldots, \#(t(p))\}$.

A symbol of arity zero is also called a *constant*. A finite tree over a ranked alphabet is also called a *term*.

First Order Logic

Syntax

The *alphabet* of FOL consists of the following symbols:

- *predicate symbols*: $p_1, p_2, \ldots, =$
- *function symbols*: $f_1, f_2, \ldots$
- *constant symbols*: $c_1, c_2, \ldots$
- *first-order variables*: $x, y, z, \ldots$
- *connectives*: $\vee, \wedge, \rightarrow, \leftrightarrow, \neg, \perp, \forall, \exists$

Syntax

The set of *first-order terms* is defined inductively:

- any constant symbol c is a term,
- any first-order variable x is a term,
- if $t_1, t_2, \ldots, t_n$ are terms and f is a function symbol of arity $n > 0$, then $f(t_1, t_2, \ldots, t_n)$ is a term,
- nothing else is a term.

A term with no variable is said to be a *ground term*.

Syntax

The set of *first-order formulae* is defined inductively:

- $\bot$ and $\top$ are formulae,
- if $t_1, t_2, \ldots, t_n$ are terms and p is a predicate symbol of arity $n > 0$, then $p(t_1, t_2, \ldots, t_n)$ is a formula,
- if t_1, t_2 are terms, then $t_1 = t_2$ is a formula,
- if φ and ψ are formulae, then $\varphi \bullet \psi$, $\neg\varphi$, $\forall x \,.\, \varphi$ and $\exists x \,.\, \varphi$ are formulae, for $\bullet \in \{\vee, \wedge, \to, \leftrightarrow\}$,
- nothing else is a formula.

An *atomic proposition* is any formula $p(t_1, \ldots, t_n)$ or $t_1 = t_2$, where p is a predicate symbol and $t_1, t_2, \ldots, t_n$ are terms.

The *language* of logic FOL is the set of formulae, denoted as $\mathcal{L}(FOL)$.

FOL Formulae

$$x = y$$

$$\forall x \forall y \, . \, x = y \leftrightarrow y = x$$

$$\forall x (\exists y \, . \, p(x, y)) \rightarrow q(x)$$

$$\forall x \, . \, p(x) \rightarrow q(f(x))$$

$$\forall x \exists y \, . \, f(x) = y \wedge (\forall z \, . \, f(z) = y \rightarrow z = x)$$

FOL Formulae

The *size* of a formula is the number of subformulae it contains, in other words, the number of nodes in the syntax tree representing the formula. The size of φ is denoted as $|\varphi|$.

The variables within the scope of a quantifier are said to be *bound*. The variables that are not bound are said to be *free*. We denote by $FV(\varphi)$ the set of free variables in φ. If $FV(\varphi) = \emptyset$ then φ is said to be a *sentence*.

Example 1 $FV(\forall x\ .\ x = y \wedge x = z \rightarrow p(x)) = \{y, z\}\square$

If $x \in FV(\varphi)$, we denote by $\varphi[x/t]$ the formula obtained from φ by substituting x with the term t.

Semantics

A *structure* is a tuple $\mathfrak{m} = \langle U, \bar{p_1}, \bar{p_2}, \ldots, \bar{f}_1, \bar{f}_2, \ldots \rangle$, where:

- U is a (possible infinite) set called the *universe*,
- $\bar{p_i} \subseteq U^{\#(p_i)}$, $i = 1, 2, \ldots$ are the *predicates*,
- $\bar{f}_i : U^{\#(f_i)} \rightarrow U$, $i = 1, 2, \ldots$ are the *functions*,

The elements of the universe are called *individuals*, denoted by $\bar{c_1}, \bar{c_2}, \ldots$.

NB: Every constant c from the alphabet of FOL has a corresponding individual $\bar{c}$, but not viceversa.

The symbol 0 has a corresponding number $\bar{0} \in \mathbb{N}$, and the function symbol s has a corresponding function $x \mapsto x + 1$. The number $\bar{1} \in \mathbb{N}$ is denoted as $s(0)$, the number $\bar{2} \in \mathbb{N}$ as $s(s(0))$, etc.

Semantics

Let $\mathfrak{m} = \langle U, \bar{p_1}, \bar{p_2}, \ldots, \bar{f_1}, \bar{f_2}, \ldots \rangle$ be a *structure*.

The interpretation of variables is a function:

$$\iota : \{x, y, z, \ldots\} \to U$$

The interpretation function is extended to terms t, denoted as $\iota(t) \in U$:

$$\begin{array}{rcl} \iota(c) & = & \bar{c} \\ \iota(f(t_1, \ldots, t_n)) & = & \bar{f}(\iota(t_1), \ldots, \iota(t_n)) \end{array}$$

Semantics

The *meaning of a sentence φ in the structure $\mathfrak{m}$ under the interpretation ι* is denoted as $[\![\varphi]\!]^{\mathfrak{m}}_{\iota} \in \{\mathsf{true}, \mathsf{false}\}$:

$$
\begin{array}{ccllll}
[\![\bot]\!]^{\mathfrak{m}}_{\iota} & = & \mathsf{false} & & & \\
[\![p(t_1, \ldots, t_n)]\!]^{\mathfrak{m}}_{\iota} & = & \mathsf{true} & \mathsf{iff} & \langle \iota(t_1), \ldots, \iota(t_n) \rangle \in \bar{p} & \\
[\![t_1 = t_2]\!]^{\mathfrak{m}}_{\iota} & = & \mathsf{true} & \mathsf{iff} & \iota(t_1) = \iota(t_2) & \\
[\![\neg\varphi]\!]^{\mathfrak{m}}_{\iota} & = & \mathsf{true} & \mathsf{iff} & [\![\varphi]\!]^{\mathfrak{m}}_{\iota} = \mathsf{false} & \\
[\![\varphi \wedge \psi]\!]^{\mathfrak{m}}_{\iota} & = & \mathsf{true} & \mathsf{iff} & [\![\varphi]\!]^{\mathfrak{m}}_{\iota} = [\![\psi]\!]^{\mathfrak{m}}_{\iota} = \mathsf{true} & \\
[\![\exists x \,.\, \varphi]\!]^{\mathfrak{m}}_{\iota} & = & \mathsf{true} & \mathsf{iff} & [\![\varphi]\!]^{\mathfrak{m}}_{\iota[x \leftarrow u]} = \mathsf{true} & \text{for some } u \in U
\end{array}
$$

where $\iota[x \leftarrow u](y) = \iota(y)$ if $x \neq y$ and $\iota[x \leftarrow u](x) = u$.

Semantics

Derived meanings:

$$
\begin{aligned}
[\![\varphi \vee \psi]\!]_\iota^{\mathfrak{m}} &= [\![\neg(\neg\varphi \wedge \neg\psi)]\!]_\iota^{\mathfrak{m}} \\
[\![\varphi \rightarrow \psi]\!]_\iota^{\mathfrak{m}} &= [\![\neg\varphi \vee \psi]\!]_\iota^{\mathfrak{m}} \\
[\![\varphi \leftrightarrow \psi]\!]_\iota^{\mathfrak{m}} &= [\![(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)]\!]_\iota^{\mathfrak{m}} \\
[\![\forall x \,.\, \varphi]\!]_\iota^{\mathfrak{m}} &= [\![\neg\exists x \,.\, \neg\varphi]\!]_\iota^{\mathfrak{m}}
\end{aligned}
$$

Decision Problems

If $FV(\varphi) = \emptyset$ we denote the meaning of φ in $\mathfrak{m}$ by $[\![\varphi]\!]^{\mathfrak{m}}$ (the choice of ι is irrelevant)

If $[\![\varphi]\!]^{\mathfrak{m}} =$ true we say that $\mathfrak{m}$ is a *model* of φ, denoted as $\mathfrak{m} \models \varphi$.

If $\mathfrak{m} \models \varphi$ for all structures $\mathfrak{m}$, we say that φ is *valid*, denoted as $\models \varphi$.

If φ has at least one model, we say that it is *satisfiable*.

Satisfiability: Given φ is it satisfiable?

Model Checking: Given $\mathfrak{m}$ and φ, does $\mathfrak{m} \models \varphi$?

Examples

Let $\leq$ be a binary predicate symbol, and $\mathfrak{m} = \langle U, \bar{\leq} \rangle$ be a structure. $\mathfrak{m}$ is a partially ordered set if $\mathfrak{m} \models \varphi_1 \wedge \varphi_2$, where:

$$\varphi_1 \;:\; \forall x \forall y \,.\, x \leq y \wedge y \leq x \leftrightarrow x = y$$

$$\varphi_2 \;:\; \forall x \forall y \forall z \,.\, x \leq y \wedge y \leq z \rightarrow x \leq z$$

Notice that $\models \varphi_1 \rightarrow \forall x \,.\, x \leq x$.

$\mathfrak{m}$ is a linearly ordered set if $\mathfrak{m} \models \varphi_1 \wedge \varphi_2 \wedge \varphi_3$, where:

$$\varphi_3 \;:\; \forall x \forall y \,.\, x \leq y \vee y \leq x$$

Exercises

Exercise 1 *Two problems P and Q are equivalent when a method for solving P is also a method for solving Q, and viceversa. Show that satisfiability and validity of first-order sentences are equivalent problems.* $\square$

Exercise 2 *Prove the validity of the following sentences:*

$$\forall x \forall y \forall z \, . \, x = y \wedge y = z \rightarrow x = z$$

$$(\exists x \, . \, \varphi \vee \psi) \leftrightarrow ((\exists x \, . \, \varphi) \vee (\exists x \, . \, \psi))$$

$$(\forall x \, . \, \varphi \wedge \psi) \leftrightarrow ((\forall x \, . \, \varphi) \wedge (\forall x \, . \, \psi))$$

$$(\exists x \, . \, \varphi \wedge \psi) \rightarrow ((\exists x \, . \, \varphi) \wedge (\exists x \, . \, \psi))$$

$$\neg(((\exists x \, . \, \varphi) \wedge (\exists x \, . \, \psi)) \rightarrow (\exists x \, . \, \varphi \wedge \psi))$$

$$((\forall x \, . \, \varphi) \vee (\forall x \, . \, \psi)) \rightarrow (\forall x \, . \, \varphi \vee \psi)$$

$$\neg((\forall x \, . \, \varphi \vee \psi) \rightarrow ((\forall x \, . \, \varphi) \vee (\forall x \, . \, \psi)))$$

Normal Forms

A formula $\varphi \in \mathcal{L}(FOL)$ is said to be *quantifier-free* iff it contains no quantifiers.

A quantifier-free formula $\varphi \in \mathcal{L}(FOL)$ is said to be in *negation normal form* (NNF) iff the only subformulae appearing under negation are atomic propositions.

A formula $\varphi \in \mathcal{L}(FOL)$ is said to be in *prenex normal form* (PNF) iff

$$\varphi = Q_1 x_1 Q_2 x_2 \ldots Q_n x_n \; . \; \psi(x_1, x_2, \ldots, x_n)$$

where $Q_i \in \{\exists, \forall\}$ and ψ is a quantifier-free formula. Sometimes ψ is said to be the *matrix* of φ.

Normal Forms

A quantifier-free formula $\varphi \in \mathcal{L}(FOL)$ is said to be in *disjunctive normal form* (DNF) iff

$$\varphi = \bigvee_i \bigwedge_j \lambda_{ij}$$

where λ_{ij} are either atomic propositions or negations of atomic propositions.

A quantifier-free formula $\varphi \in \mathcal{L}(FOL)$ is said to be in *conjunctive normal form* (CNF) iff

$$\varphi = \bigwedge_i \bigvee_j \lambda_{ij}$$

where λ_{ij} are either atomic propositions or negations of atomic propositions.

FOL on Finite Words

Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet and $w : \{0, 1, \ldots, n-1\} \to \Sigma$ be a finite word, i.e. $w = a_0 a_1 \ldots a_{n-1} \in \Sigma^*$.

The structure corresponding to w is $\mathfrak{m}_w = \langle dom(w),\ \{\bar{p_a}\}_{a \in \Sigma},\ \bar{\leq} \rangle$, where:

- $dom(w) = \{0, 1, \ldots, n-1\}$,
- $\bar{p_a} = \{x \in dom(w) \mid w(x) = a\}$,
- $x \bar{\leq} y$ iff $x \leq y$.

$$\mathfrak{m}_{abbaab} = \langle \{0, \ldots, 5\},\ \bar{p_a} = \{0, 3, 4\},\ \bar{p_b} = \{1, 2, 5\},\ \bar{\leq} \rangle$$

Exercises

Exercise 3 *Write a FOL formula* $S(x, y)$ *which is valid for all positions* $x, y \in \mathbb{N}$ *such that* $y = x + 1$. $\square$

Exercise 4 *Write a FOL sentence whose models are all words with* a *on even positions and* b *on odd positions. Next, (try to) write a FOL sentence whose models are all words with* a *on even positions.* $\square$

Exercise 5 *Write a FOL formula* $len(x)$ *that is satisfied by all words of length* x. $\square$

Exercise 6 *Write a FOL sentence whose models are all finite words.* $\square$

FOL on Infinite Words

Let $w : \mathbb{N} \to \Sigma$ be an infinite word.

The structure corresponding to w is $\mathfrak{m}_w = \langle \mathbb{N}, \{\bar{p_a}\}_{a \in \Sigma}, \bar{\leq} \rangle$.

$\mathfrak{m}_{(ab)^\omega} = \langle \mathbb{N}, \bar{p_a} = \{2k \mid k \in \mathbb{N}\}, \bar{p_b} = \{2k+1 \mid k \in \mathbb{N}\}, \bar{\leq} \rangle$

FOL on Finite Trees

Let $\Sigma = \{f, g, \ldots\}$ be an alphabet and $t : \mathbb{N}^* \mapsto \Sigma$ be a finite tree over Σ.

The structure corresponding to t is $\mathfrak{m}_t = \langle dom(t), \{\bar{p_f}\}_{f \in \Sigma}, \preceq, \{s_n\}_{n \in \mathbb{N}} \rangle$, where:

- $\bar{p_f} = \{p \in dom(t) \mid t(p) = f\}$,
- $\preceq$ is the prefix order on $\mathbb{N}^*$,
- $s_n(p) = \begin{cases} pn, \text{ if } pn \in dom(t) \\ p, \text{ otherwise} \end{cases}$ for all $n \in \mathbb{N}$, is the n-th successor.

Examples

$\mathfrak{m}_{f(f(g,g),g)} = \langle\{\epsilon, 0, 1, 00, 01, 10, 11\}, \bar{p_f} = \{\epsilon, 0, 1\}, \bar{p_g} = \{00, 01, 10, 11\}, \bar{\lesssim}, \{s_0, s_1\}\rangle$, where:

- $s_i(p) = pi$, for all $p \in \{\epsilon, 0, 1\}$ and $i \in \{0, 1\}$,
- $s_0(00) = s_1(00) = 00$, $s_0(01) = s_1(01) = 01$, $s_0(10) = s_1(10) = 10$ and $s_0(11) = s_1(11) = 11$.

The *lexicographic order* on $\{0, 1\}^*$ is defined as follows:

$x \preceq_{lex} y \stackrel{\text{def}}{=} x \preceq y \vee \exists z \,.\, s_0(z) \prec x \wedge s_1(z) \prec y$, where $x \prec y \stackrel{\text{def}}{=} x \preceq y \wedge \neg(x = y)$

Exercise 7 *Write a formula that defines all nodes on a path of a binary tree.* $\square$

Exercise 8 *A red-black tree is a tree in which all nodes are either red or black, such that the root is black, and each red node has only black children. Write a FOL sentence whose models are all red-black trees.* $\square$

FOL on Infinite Trees

Let $t : \mathbb{N}^* \mapsto \Sigma$ be an infinite tree over Σ.

The structure corresponding to t is $\mathfrak{m}_t = \langle \mathbb{N}^*, \{\bar{p_f}\}_{f \in \Sigma}, \preceq, \{s_n\}_{n \in \mathbb{N}} \rangle$, where:

- $\bar{p_f} = \{p \in \mathbb{N}^* \mid t(p) = f\}$,
- $\preceq$ is the prefix order on $\mathbb{N}^*$,
- $s_n(p) = pn$, for all $n \in \mathbb{N}$, is the n-th successor.

Exercise 9 *Given a (possibly infinite) set $\mathcal{T} = \{t_1, t_2, \ldots\}$ of finite or infinite trees, of finite or infinite branching degrees, represent each tree $t_i \in \mathcal{T}$ as an infinite binary tree $\bar{t}_i : \{0,1\}^* \to \Sigma$.* $\square$

Monadic Second Order Logic

Syntax

The alphabet of MSOL consists of:

- all first-order symbols
- *set variables*: $X, Y, Z, \ldots$

The set of MSOL terms consists of all first-order terms and set variables. The set of MSOL formulae consists of:

- all first-order formulae, i.e. $\mathcal{L}(FOL) \subseteq \mathcal{L}(MSOL)$,
- if t is a term and X is a set variable, then $X(t)$ is a formula,
- if φ and ψ are formulae, then $\varphi \bullet \psi$, $\neg\varphi$, $\forall x \,.\, \varphi$, $\exists x \,.\, \varphi$, $\forall X \,.\, \varphi$ and $\exists X \,.\, \varphi$ are formulae, for $\bullet \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$.

$X(t)$ is sometimes written $t \in X$.

Examples

Universal set:

$$\forall x \,.\, X(x)$$

$X \subseteq Y$:

$$\forall x \,.\, X(x) \to Y(x)$$

$X \neq Y$:

$$\exists x \,.\, (X(x) \wedge \neg Y(x)) \;\vee\; (\neg X(x) \wedge Y(x))$$

$X = \emptyset$:

$$\forall x \,.\, \neg X(x)$$

Singleton set:

$$\forall Y \,.\, ((\forall x \,.\, Y(x) \to X(x)) \wedge \exists x \,.\, X(x) \wedge \neg Y(x)) \to \forall x \,.\, \neg Y(x)$$

Semantics

Let $\mathfrak{m} = \langle U, \bar{p_1}, \bar{p_2}, \ldots, \bar{f_1}, \bar{f_2}, \ldots \rangle$ be a *structure*.

The interpretation of variables is a function:

$$\iota : \{x, y, z, \ldots\} \cup \{X, Y, Z, \ldots\} \to U \ \cup \ 2^U$$

such that:

- $\iota(x) \in U$ for each individual variable x
- $\iota(X) \in 2^U$ for each set variable X

$$[\![\exists X \ . \ \varphi]\!]^{\mathfrak{m}}_{\iota} \quad = \quad \text{true} \quad \text{iff} \quad [\![\varphi]\!]^{\mathfrak{m}}_{\iota[X \leftarrow S]} = \text{true} \quad \text{for some } S \subseteq U$$

MSOL Example

Example 2 *The MSOL formula that characterizes all partitions* $\langle X, Y \rangle$ *of* Z*:*

$$partition(X, Y, Z) \; : \; (\forall x \forall y \, . \, X(x) \wedge Y(y) \rightarrow \neg x = y) \wedge (\forall x \, . \, Z(x) \leftrightarrow X(x) \vee Y(x))$$

$\square$

MSOL on Words: (W)S1S

Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet. The alphabet of the sequential calculus is composed of:

- the function symbol s denotes the successor,
- the set constants $\{p_a \mid a \in \Sigma\}$; p_a denotes the set of positions of a
- the first and second order variables and connectives.

(W)eak indicates that quantification is over finite sets only.

Example 3 **Q**: *Let* $\mathfrak{m}_{abbaab} = \langle\{0, \ldots, 5\}, \bar{p_a} = \{0, 3, 4\}, \bar{p_b} = \{1, 2, 5\}, \bar{s}\rangle$ *be a finite word, where* $\bar{s}(n) = n + 1$*, for* $n = 0, \ldots, 4$ *and* $\bar{s}(5) = 5$*.* $\square$

Examples

The order $x \leq y$ on positions is defined as:

- $closed(X) \ : \ \forall x \ . \ X(x) \to X(s(x))$
- $x \leq y \ : \ \forall X \ . \ X(x) \wedge closed(X) \to X(y)$

The set of positions of a word is defined by $pos(X) : \forall x \ . \ X(x)$.

Examples

The first position is: $zero(x) \ : \ \forall y \ . \ x \leq y$

The set of even positions is defined by

$$\begin{aligned} even(X) \ &: \ \exists z \ . \ zero(z) \wedge X(z) \wedge \\ & \exists Y \exists Z \ . \ pos(Z) \wedge partition(X, Y, Z) \wedge \\ & \forall x \forall y \ . \ X(x) \wedge \neg s(x) = x \rightarrow Y(s(x)) \wedge \\ & \forall x \forall y \ . \ Y(x) \wedge \neg s(x) = x \rightarrow X(s(x)) \end{aligned}$$

The set of all words having a's on even positions is the set of models of the sentence: $\exists X \ . \ even(X) \wedge \forall x \ . \ X(x) \rightarrow p_a(x)$

Exercise 10 *Write a S1S formula whose models are exactly all infinite words starting with an even number of 0's followed by an infinite number of 1's.* $\square$

MSOL on Trees: (W)SkS

Let $\Sigma = \{a, b, \ldots\}$ be a tree alphabet. The alphabet of (W)SkS is:

- the function symbols $\{s_i \mid i = 1, \ldots, k\}$, where $s_i(x)$ denotes the i-th successor of x; if we allow $\{s_i \mid i \in \mathbb{N}\}$, the logic is called (W)SωS,
- the predicate symbols $\{p_a \mid a \in \Sigma\}$; p_a denotes the set of positions of a
- the first and second order variables and connectives.

In FOL on trees we had $\leq$ (prefix) instead of s_i. Why ?

Examples

Let us consider binary trees, i.e. the alphabet of S2S.

- The formula $closed(X) \ : \ \forall x \ . \ X(x) \rightarrow X(s_0(x)) \wedge X(s_1(x))$ denotes the fact that X is a downward-closed set.
- The prefix ordering on tree positions is defined by $x \leq y \ : \ \forall X \ . \ closed(X) \wedge X(x) \rightarrow X(y)$.
- The root of a tree is defined by $root(x) \ : \ \forall y \ . \ x \leq y$.

Exercise

Exercise 11 *Define the set of binary trees* $t : \{0,1\}^* \to \{a,b\}$ *such that* $t(p) = a$ *if* p *is of even length.* $\square$

Exercise 12 *Write a S2S formula* $path(X)$ *that defines the set of all paths in a binary tree.* $\square$

Exercise 13 *Write a S2S sentence whose models are all finite trees.* $\square$