Obligation and Weak-Parity Games

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Grenoble, December 2018

Hierarchy

4. Reactivity: Muller, Parity

3. Recurrence: Büchi   Persistence: co-Büchi

2. Obligation: Staiger-Wagner, Weak-Parity

1. Reachability   Safety

Obligation Games

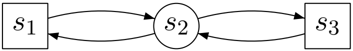
We consider games where the winning condition for Player 0 (on the play) is

- a Boolean combination of reachability conditions
- equivalently: a condition on the set Occ

Standard form: Staiger-Wagner winning condition, using
$F = \{F_1, \ldots, F_k\}$
Player 0 wins play ρ iff $\mathrm{Occ}(\rho) \in F$. We call these games obligation games (or Staiger-Wagner games).

Example

$S = \{s_1, s_2, s_3\}$ $F = \{\{s_1, s_2, s_3\}\}$



No winning strategy is positional.

There is a finite-state winning strategy.

Weak Parity Games

Method for solving Staiger-Wagner games:

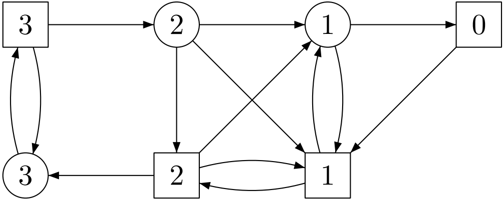
1. Solve weak parity games.
2. Reduce Staiger-Wagner games to weak parity games.

A weak parity game is a pair (G, p), where

- $G = (S, S_0, E)$ is a game graph and
- $p : S \to \{0, \ldots, k\}$ is a priority function mapping every state in S to a number in $\{0, \ldots, k\}$.

A play ρ is winning for Player 0 iff the minimum priority occurring in ρ is even: $\min_{s \in \mathrm{Occ}(\rho)} p(s)$ is even

Example



Weak Parity Games

Theorem

For a weak parity game one can compute the winning regions W_0, W_1 and also construct corresponding positional winning strategies.

Proof.

Let $G = (S, S_0, E)$ be a game graph, $p : S \to \{0, \dots, k\}$ a priority function. Let $P_i = \{s \in S \mid p(s) = i\}$.

First steps if $P_0 \neq \emptyset$: We first compute $A_0 = \mathrm{Attr}_0(P_0)$, clearly from here Player 0 can win.

In the rest game, we compute $A_1 = \mathrm{Attr}_1(P_1 \setminus A_0)$ from here Player 1 can win.

General Construction

Aim: Compute A_0, $A_1, \ldots A_k$

Let G_i be the game graph restricted to $S \setminus (A_0 \cup \ldots A_{i-1})$.

$\mathrm{Attr}_0^{G_i}(M)$ is the 0-attractor of M in the subgraph induced by G_i

$$
\begin{aligned}
A_0 \quad & := \mathrm{Attr}_0(P_0) \\
A_1 \quad & := \mathrm{Attr}_1^{G_1}(P_1 \setminus A_0) \\
\text{for } i > 1 : & \\
A_i \quad & := \begin{cases} \mathrm{Attr}_0^{G_i}(P_i \setminus (A_0 \cup .. \cup A_{i-1})) & \text{if } i \text{ is even} \\ \mathrm{Attr}_1^{G_i}(P_i \setminus (A_0 \cup .. \cup A_{i-1)}) & \text{if } i \text{ is odd} \end{cases}
\end{aligned}
$$

Correctness

Correctness Claim:

$$W_0 = \bigcup_{i \text{ even}} A_i \text{ and } W_1 = \bigcup_{i \text{ odd}} A_i$$

and the union of the corresponding attractor strategies are positional winning strategies for the two players on their respective winning regions.

Prove by induction on $j = 0, \ldots, k$ the following:

$$\bigcup_{i=0..j, i \text{ even}} A_i \subseteq W_0 \text{ and } \bigcup_{i=1..j, i \text{ odd}} A_i \subseteq W_1$$

Correctness (cont.)

Base:

- i=0: $A_0 = \mathrm{Attr}_0(P_0) \subseteq W_0$
- i=1: $A_1 = \mathrm{Attr}_1(P_1 \setminus A_0) \subseteq W_1$

Induction step:

- i even: Consider play ρ starting A_i that complies to attractor strategy.
  - Case 1: ρ eventually leaves A_i to some A_j (from a Player-1 state), which $j < i$ and even, then Player 0 wins by induction hypothesis.
  - Case 2: ρ visits P_i, then we need to show that ρ visits only states with $p(s) \geq i$. Consider a state s that visits P_i, then
    - if $s \in S_0$, then not all edges lead to states with lower priority, otherwise $s \in A_j$ for some $j < i$. Contradiction.

Correctness (cont.)

- Case 2 (cont.):
  - if $s \in S_1$, then all edges lead to states with priority $\geq i$. Any edge to a lower priority must lead to A_j with even j (Case 1). If there were edges to states s' with priority $j < i$ and j odd, then s' would already be in A_j. Contradiction.
- i odd: switch players

Obligation/Staiger-Wagner to Weak-Parity Games

- How to translate a Staiger-Wagner automaton to Weak-Parity automaton?
- Idea: record visited states during a run
- Record set: $R \subseteq S$
- Question: How to give priorities?

Record Sets and Priorities

Assume automaton with states $\{s_0, s_1, s_2\}$.

Consider possible record sets:

$$
\begin{tikzcd}
 & \{s_0, s_1, s_2\} & \\
\{s_0, s_1\} \arrow[ur] & \{s_0, s_2\} \arrow[u] & \{s_1, s_2\} \arrow[ul] \\
\{s_0\} \arrow[u] \arrow[ur] & \{s_1\} \arrow[ul] \arrow[ur] & \{s_2\} \arrow[ul] \arrow[u] \\
 & \emptyset \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

Assume the following run $s_1, s_0, s_1, s_0, s_2, \ldots$ and the acceptance condition $F = \{\{s_0, s_1\}, \{s_0, s_1, s_2\}\}$. How to assign priorities?

Record Sets and Priorities

$F = \{\{s_0, s_1\}, \{s_0, s_1, s_2\}\}$. How would you assign priorities?

$$
\begin{tikzcd}
 & & \{s_0, s_1, s_2\} : 0 & & & \text{0 or 1} \\
\{s_0, s_1\} : 2 \arrow[urr] & & \{s_0, s_2\} : 3 \arrow[u] & & \{s_1, s_2\} : 3 \arrow[ull] & \text{2 or 3} \\
\{s_0\} : 5 \arrow[u] \arrow[urr] & & \{s_1\} : 5 \arrow[ull] \arrow[urr] & & \{s_2\} : 5 \arrow[ull] \arrow[u] & \text{4 or 5} \\
 & & \emptyset \arrow[ull] \arrow[u] \arrow[urr] & & & \text{d.c.}
\end{tikzcd}
$$

From Staiger-Wagner to Weak Parity Automata

Given a deterministic Staiger-Wagner automaton $A = (S, I, T, F)$, we can construct an equivalent weak parity automaton $A' = (S', I', T', p)$ as follows:

$$
\begin{aligned}
S' \quad & := S \times 2^S \\
I' \quad & := (I, \{I\}) \\
T'((s, R), a) \quad & := (T(s, a), R \cup \{T(s, a)\} \\
p((s, R)) \quad & := 2 \cdot |S| - \begin{cases} 2 \cdot |R| & \text{if } R \in F \\ 2 \cdot |R| - 1 & \text{if } R \notin F \end{cases}
\end{aligned}
$$

Idea of Game Reduction

We want to solve Staiger-Wagner games. We use a reduction to weak parity games (and the positional winning strategies of weak parity games).

Reduction will transform a game (G, ϕ) into a game (G', ϕ') such that usually

- G' is (usually) larger than G
- ϕ' is simpler than ϕ (so the solution of (G', ϕ') is simpler than that of (G, ϕ))
- from a solution of (G', ϕ') we can construct a solution of (G, ϕ).

Concrete application: Transform Staiger-Wagner game into a weak parity game over a larger graph (from S proceed to $S \times 2^S$)

Game Reduction

Let $G = (S, S_0, E)$ and $G' = (S', S'_0, E')$ be game graphs with winning conditions ϕ and ϕ', respectively.

(G, ϕ) is reducible to (G', ϕ') if:

1. $S' = S \times M$ for a finite set M and $S'_0 = S_0 \times M$
2. Each play $\rho = s_0 s_1 \dots$ over G is translated into a play $\rho' = s'_0 s'_1 \dots$ over G' by
   - a function $f : S \to S \times M$ (the beginning of ρ').
   - forall states $(m, s) \in S \times M$ in G' and all states $s' \in S$ in G, if there exists an edge $(s, s') \in E$, then there is a unique m' with $((m, s), (m', s')) \in E'$
   - forall edges $((m, s), (m', s')) \in E'$ in G', there is an edges $(s, s') \in E$ in G
3. For all plays ρ and ρ' according to 2.: $\rho \in \phi$ iff $\rho' \in \phi'$

Application of Game Reduction

Theorem

Suppose (G, ϕ) is reducible to (G', ϕ') with extension set M, initial function g, and G and G' defined as before. Then, if Player 0 wins in (G', ϕ') from $g(s)$ with a memoryless winning strategy, then Player 0 wins in (G, ϕ) from s with a finite-state strategy.

Idea: Given a memoryless winning strategy $f : S'_0 \to S'$ from $g(s)$ for Player 0 in (G', ϕ'), we can construct a strategy automaton $A = (M, m_0, \delta, \lambda)$ for Player 0 in (G, ϕ).

Obligation/Staiger-Wagner Games

Theorem

Given a Staiger-Wagner game (G, ϕ), one can compute the winning regions of Player 0 and 1 and corresponding finite state strategies.

Proof.

We can apply game reduction with (G', ϕ') as follows:

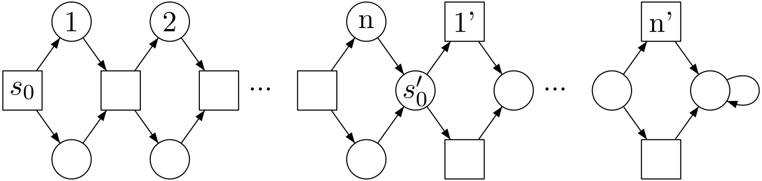
$$
\begin{aligned}
G' &:= (S', S'_0, E') \\
S' &:= 2^S \times S \\
((R, s), (R', s')) \in E') &\quad \text{iff } (s, s') \in E, R' = R \cup \{s'\} \\
g(s) &= (\{s\}, s) \\
p((R, s)) &:= 2 \cdot |S| - \begin{cases} 2 \cdot |R| & \text{if } P \in \phi \\ 2 \cdot |R| - 1 & \text{if } P \notin \phi \end{cases}
\end{aligned}
$$

Exponential-Size Memory

Theorem

There is a family of Staiger-Wagner games over game graphs $G_1, G_2, G_3, \ldots$ which grow linearly in n such that

- *Player 0 wins from a certain initial vertex of G_n*
- *any finite-state strategy for Player 0 needs at least 2^n states*



Winning condition:

$$\phi = \{\rho \mid \forall i = 1 \ldots n : i \in \mathrm{Occ}(\rho) \leftrightarrow i' \in \mathrm{Occ}(\rho)\}$$

Exponential Memory (cont.)

Claim:

Over G_n there is an automaton winning strategy for Player 0 from vertex s_0 with a memory of size 2^n. (Remember the visited vertices i, for the appropriate choice from vertex s_0' onwards.)

Each automaton winning strategy for Player 0 from s_0 in G_n has a memory of 2^n many states.

Proof.

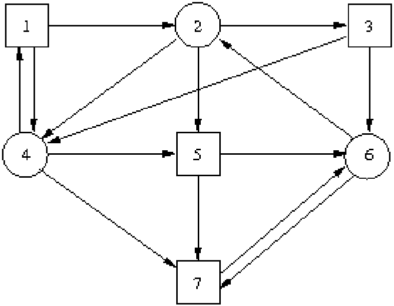
Assume $|\text{states}| < 2^n$ is sufficient.

Then two play prefixes $u \neq v$ exist leading to the same memory states at s_0'. The rest r of the play is then the same after u and v.

One of the two player ur, vr is lost by Player 0. Contradiction.

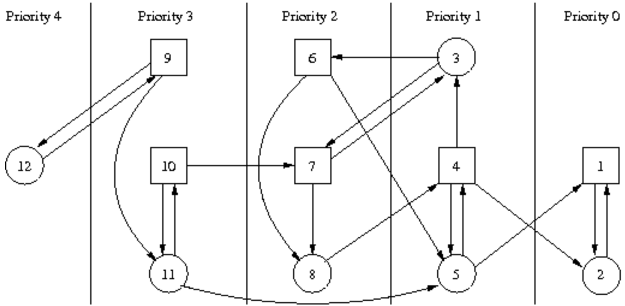
Exercise

1. Consider the game graph shown below. Let the winning condition for Player 0 be $\mathrm{Occ}(\rho) = \{1, 2, 3, 4, 5, 6, 7\}$.
    1. Find the winning region for Player 0 and describe a winning strategy
    2. Show that there is no positional winning strategy for Player 0.



Exercise

2. Compute the winning regions and the corresponding positional winning strategies for Player 0 and 1 in this weak-parity game.



Exercise

3. A winning strategy is called *uniform* if it is a winning strategy from every winning state in the game. Let (G, p) be a weak parity game and let W_0 be the winning region of Player 0. For all $s \in W_0$ let f_s be a positional winning strategy from s for Player 0. Construct a uniform winning strategy f from the strategies f_s meaning that for every $s \in W_0$ there is a $t \in W_0$, s.t. $f(s) = f_t(s)$.