About Games and Trees

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**Recap**

Winning conditions are defined over Occ and Inf.

<table>
<thead>
<tr>
<th>Occ((\rho))</th>
<th>Inf((\rho))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reachability/Guarantee game</td>
<td>Büchi game</td>
</tr>
<tr>
<td>Safety game</td>
<td>co-Büchi game</td>
</tr>
<tr>
<td>Weak-parity game</td>
<td>Parity game</td>
</tr>
<tr>
<td>Obligation/Staiger-Wagner game</td>
<td>Muller game</td>
</tr>
<tr>
<td></td>
<td>LTL games</td>
</tr>
</tbody>
</table>
## Recap

How did we solve those games?

<table>
<thead>
<tr>
<th>Game</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reachability games</td>
<td>Attractor + Attractor Strategy</td>
</tr>
<tr>
<td>Safety games</td>
<td>like Reachability games</td>
</tr>
<tr>
<td>Büchi games</td>
<td>Recurrence set + Extended Attractor Strategy</td>
</tr>
<tr>
<td>co-Büchi games</td>
<td>like Büchi games</td>
</tr>
<tr>
<td>Weak-parity games</td>
<td>Alternation between $\text{Attr}_0$ and $\text{Attr}_1$</td>
</tr>
<tr>
<td>Obligation games</td>
<td>Reduction to Weak-parity games with record sets</td>
</tr>
<tr>
<td>Parity games</td>
<td>Recursive algorithm, Progress-Measure algorithm, and</td>
</tr>
<tr>
<td></td>
<td>Strategy Improvement algorithm</td>
</tr>
</tbody>
</table>
Muller, Rabin, and Streett Games
Muller Games

Given a game graph $G = (S, S_0, E)$ and a Muller condition $\mathcal{F} \subseteq \mathcal{P}(S)$, then a play $\rho$ is winning for Player 0 if exists $F \in \mathcal{F}$ s.t.

$$\text{Inf}(\rho) = F.$$

Recall, in Staiger-Wagner games, we had $\text{Occ}(\rho) = F$. 
Example

Player 0 wins iff the number of states in $S_0 = \{s_1, s_2, s_3, s_4\}$ visited infinitely often is equal to the lowest index of the states in $S_1 = \{t_1, t_2, t_3, t_4\}$ visited infinitely often.

![Diagram](image)

Winning condition in Muller form: $F \in \mathcal{F}$ iff $\min_i (t_i \in F) = |F \cap S_0|$. 
Record the Past

For simplicity, we record only the $s$-states.

<table>
<thead>
<tr>
<th>Visited letter</th>
<th>Record set</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>$s_1$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>$s_1s_3$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>$s_1s_3$</td>
</tr>
<tr>
<td>$s_4$</td>
<td>$s_1s_3s_4$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$s_1s_2s_3s_4$</td>
</tr>
<tr>
<td>$s_4$</td>
<td>$s_1s_2s_3s_4$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>–”–</td>
</tr>
<tr>
<td>$s_4$</td>
<td>–”–</td>
</tr>
<tr>
<td>$s_4$</td>
<td>–”–</td>
</tr>
</tbody>
</table>
## Latest Appearance Record

<table>
<thead>
<tr>
<th>Visited letter</th>
<th>Record set</th>
<th>LAR</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>$s_1$</td>
<td>$s_1 s_2 s_3 s_4(1)$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>$s_1 s_3$</td>
<td>$s_3 s_1 s_2 s_4(3)$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>$s_1 s_3$</td>
<td>$s_3 s_1 s_2 s_4(1)$</td>
</tr>
<tr>
<td>$s_4$</td>
<td>$s_1 s_3 s_4$</td>
<td>$s_4 s_3 s_1 s_2(4)$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$s_1 s_2 s_3 s_4$</td>
<td>$s_2 s_4 s_3 s_1(4)$</td>
</tr>
<tr>
<td>$s_4$</td>
<td>$s_1 s_2 s_3 s_4$</td>
<td>$s_4 s_2 s_3 s_4(2)$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>-”-”-</td>
<td>..</td>
</tr>
<tr>
<td>$s_4$</td>
<td>-”-”-</td>
<td>..</td>
</tr>
<tr>
<td>$s_4$</td>
<td>-”-”-</td>
<td>..</td>
</tr>
</tbody>
</table>
Example

Assume the states $s_3$ and $s_4$ are repeated infinitely often but not $s_1, s_2$. Then:

- the states $s_1$ and $s_2$ eventually arrive at the last two positions and are not touched any more, so finally the hit appears at most on positions 1 and 2

- position 2 is hit again and again; if only position 1 is hit from some point onwards, only the same letter would be chosen from there onwards (and not two states $s_3$ and $s_4$ as assumed)
Example

LAR-strategy for Player 0:

During play update and use the LAR as follows:

- shift the letter of the current state to the front
- record the position from where the current letter was taken
- move to the state whose index is the current hit position

This is a finite-state winning strategy with $n! \cdot n$ memory states if $n$ letter states and $n$ number states occur in the game graph.
Theorem

For a game \((G, \phi)\) with \(G = (S, S_0, E)\) and Muller winning condition \(\phi\) (using the set \(F \subseteq 2^S\)), there is a game \((G', \phi')\) with \(G' = (S', S'_0, E')\) and parity winning condition \(\phi'\) such that \((G, \phi) \leq (G', \phi')\)

Proof.

Assume \(= \{1, \ldots n\}\). Define \(S' := LAR(S)\)

\(LAR(S)\) is the set of pairs \(((i_1, \ldots i_n), h)\) consisting of a permutation of 1, \ldots \(n\) and a number \(h \in \{1, \ldots n\}\).
**Construction**

**Initialisation:** For $i \in S$ set

$$g(i) = ((i, i + 1, \ldots, n, 1, \ldots, i - 1), 1)$$

**Definition of $E'$:** Introduce an edge from $((i_1 \ldots i_n), h)$ to $((i_{m}i_1 \ldots i_{m-1}i_{m+1} \ldots i_n), m)$ if $(i_1, i_m) \in E$
Construction

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How should we assign the priorities?
Record Sets and Priorities

Recall, priorities in the reduction of Staiger-Wagner to Weak-Parity. 

$F = \{\{s_0, s_1\}, \{s_0, s_1, s_2\}\}$. 

$\emptyset$ 

$\{s_0\} : 5$  

$\{s_1\} : 5$  

$\{s_2\} : 5$  

$\{s_0, s_1\} : 2$  

$\{s_0, s_2\} : 3$  

$\{s_1, s_2\} : 3$  

$\{s_0, s_1, s_2\} : 0$  

0 or 1  

2 or 3  

4 or 5  

d.c.
Construction(2)

Now, we are only interested in states visited infinitely often. The hit value tells us how many states are visited infinitely often. E.g., if $s_0$ and $s_1$ are visited infinitely often, we see from some point on only the LARs: $(s_0s_1\ldots, 1), (s_0s_1\ldots, 2), (s_1s_0\ldots, 1), (s_1s_0\ldots, 2)$. If $\mathcal{F} = \{s_0, s_1\}$, then we want plays that visit only $(s_0s_1\ldots, 1)$ or $(s_1s_0\ldots, 1)$ from some point on to be losing. So, the priorities signed to $(s_0s_1\ldots, 2)$ or $(s_0s_1\ldots, 2)$ need to override the priorities of $(s_0s_1\ldots, 1)$ or $(s_1s_0\ldots, 1)$. 
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Priorities $p$: $\text{LAR}(S) \rightarrow \{1, \ldots 2n\}$

$$p((i_1 \ldots i_n, h)) = 2n - \begin{cases} 
2h - 1 & \text{if } \{i_1 \ldots i_h\} \notin \mathcal{F} \\
2h & \text{if } \{i_1 \ldots i_h\} \in \mathcal{F}
\end{cases}$$
Proof of Correctness

Lemma

Given a play \( \rho \) in \((G, \phi)\) and its counterpart \( \rho' \) in \((G', \phi')\), then \( \text{Inf}(\rho) = F \) with \(|F| = m\) iff

1. in \( \rho' \) the hit value is \( > m \) only finitely often
2. in \( \rho' \) the hit-segment is equal to \( F \) infinitely often

Proof (forward).

Let \( \text{Inf}(\rho) = F \) and \(|F| = m\). Choose \( k \) and \( k' > k \) s.t. forall \( j > k \)
\( \rho(j) \in F \) and \( \{\rho(k), \ldots, \rho(k' - 1)\} = F \).

By construction of \( \rho' \), the \( F \)-states \( F = \{i_1, \ldots, i_m\} \) are at the beginning of \( \rho'(k') \) and for every \( k'' > k' \) the hit is always \( \leq m \) (1).
Proof of Correctness

Proof (forward cont.)
For the hit equal to $m$ the hit-segment must be the set $F$. So, for (2) it suffices to show that the hit is infinitely often equal to $m$. Assume the hit is only finitely often equal to $m$, then eventually the LAR-entries $i_m, i_{m+1}, \ldots, i_n$ are not changed anymore (and so, these states are not visited anymore). Then, $|\text{Inf}(\rho)| < m$, which contradicts $\text{Inf}(\rho) = F$ with $|F| = m$.

Proof (backwards).
Assume (1) and (2) holds. It follows from (1), that the LAR-entries $i_{m+1}, \ldots, i_n$ in $\rho'$ are fixed from some point $j_0$ onwards. So, the states $i_{m+1}, \ldots, i_n$ are not visited anymore after $j_0$. From, (2) it follows that $i_{m+1}, \ldots, i_n$ are not in $F$ (i.e., $\text{Inf}(\rho) \subseteq F$).
Proof of Correctness

Proof (backwards cont.)

For $F \subseteq \text{Inf}(\rho)$, assume $q \in F$ but $q \not\in \text{Inf}(\rho)$.
Since $q \in F$ and hit-segment $= F$ infinitely often (2), we know that $q \in \text{hit-segment}$ infinitely often. Furthermore, since $|\text{hit-segment}| \leq m$ from some point on (1), it follows that from some point on the index $i$ of $q$ in the hit segment is $\leq m$. From $q \not\in \text{Inf}(\rho)$ it follows that from some point onwards $q$ can only stay in the same position in the LAR or go to the right and its final position $i$ is $> m$. Contradiction.
Example

\[ \rho \in \text{Win} \iff \{0, 2\} \subseteq \text{Inf}(\rho) \]

\[ \mathcal{F} = \{\{0, 2\}, \{0, 1, 2\}, \{0, 1, 2, 3\}\} \]
Summary

We can solve Muller games by reduction to parity games using the Last Appearance Record construction.
Summary

We can solve Muller games by reduction to parity games using the Last Appearance Record construction.

Finally, Rabin and Streett games can be viewed as Muller games.
Back to Tree Automata
Muller tree automaton

Recall, a Muller tree automaton over $\Sigma$ is $A = (S, s_0, T, F)$, where

- $S$ is a finite set of states,
- $s_0 \in S$ is an initial state,
- $T : S \times \Sigma \rightarrow 2^{S \times S}$ is a transition function
- $F \subseteq 2^S$ is the set of accepting sets.

Given an input tree $t$, a run $\pi$ of $A$ over $t$ is accepting iff for every path $\sigma$ in $t$:

$$\text{Inf}(\pi|_{\sigma}) \in F$$
Parity tree automaton

A Parity tree automaton over $\Sigma$ is $A = (S, s_0, T, p)$, where

- $S$ is a finite set of states,
- $s_0 \in S$ is an initial state,
- $T : S \times \Sigma \rightarrow 2^{S \times S}$ is a transition function,
- $p : S \rightarrow \{0, \ldots, k\}$ is a priority function.

Given an input tree $t$, a run $\pi$ of $A$ over $t$ is accepting iff for every path $\sigma$ in $t$:

$$\min_{s \in \text{Inf}(\pi|_{\sigma})} p(s) \text{ is even}$$
Example

A parity tree automaton over $\Sigma = \{a, b\}$ that recognizes all binary trees

$$
\mathcal{T} = \{ t \in \mathcal{T}^\omega(\Sigma) \mid \text{each path through } t \text{ has only finitely many } b \}
$$

- $S = \{q_a, q_b\}$
- $I = \{q_a, q_b\}$
- $T(q_a, a) = \{(q_a, q_a)\}$, $T(q_b, a) = \{(q_a, q_a)\}$
  - $T(q_a, b) = \{(q_b, q_b)\}$, $T(q_b, b) = \{(q_b, q_b)\}$
- $p(q_a) = 2$, $p(q_b) = 1$
Theorem

1. For any parity tree automaton one can construct an equivalent Muller tree automaton.
2. For any Muller tree automaton one can construct an equivalent parity tree automaton.

Proof 1.

Given a parity tree automaton $A = (S, s_0, T, p)$ keep states and transitions and define $F$ as follows:

$$F = \{ F \in 2^S \mid \min_{s \in F} p(s) \text{ is even} \}$$
Proof 2.

Copy the simulation of Muller games by parity games. Given a Muller tree automaton with state set \( S \) use for the parity tree automaton the state set \( \text{LAR}(S) \) and define the transition according to the LAR update rule.

Allow transition

\[
((s_1 \ldots s_n, i), a, (s'_1 \ldots s'_n, j), (s''_1 \ldots s''_n, k))
\]

for transition \((s_1, a, s'_1, s''_1)\) of the Muller automaton, where

- \((s'_1 \ldots s'_n, j)\) is the LAR update for a visit to \(s'_1\) and
- \((s''_1 \ldots s''_n, k)\) is the LAR update for a visit to \(s''_1\).

Define priorities as in the simulation of Muller games by parity games.
Tree Automata and Games

With any parity tree automaton \( A = (S, s_0, T, p) \) over \( \Sigma \) and any input tree \( t \in T^\omega(\Sigma) \), we can associate a parity game between

- Player Automaton and
- Player Pathfinder

with proceeds as follows:

- First, Automaton picks a transition in \( T \) (from \( s_0 \)) which matches the labels of the root of \( t \)
- Then Pathfinder decides on a direction (left or right) to proceed to a son of the root
- Then Automaton chooses again a transition for this node (and compatible with the first transition)
- Then Pathfinder reacts again by branching left or right...
Tree Automata and Games

Such a play give a sequence of transitions (and hence a sequence of states in $S$) built up along a path chosen by Pathfinder. Automaton wins the play iff the sequence of states satisfies the parity condition.

Given a parity tree automaton $A = (S, s_0, T, p)$ over $\Sigma$ and an input tree $t$, the game graph $G_{A,t} = (S_0 \cup S_1, S_0, E)$ is defined by

- $S_0 = \{ (w, t(w), s) \mid w \in \{0, 1\}^*, t(w) \in \Sigma, s \in S_0 \}$,
- $S_1 = \{ (w, t(w), \tau) \mid w \in \{0, 1\}^*, t(w) \in \Sigma, \tau \in T \}$,

and the edges relation $E$ is such that successive game positions are compatible with the transitions in $A$ on $t$.

The priority of a triple $u = (w, t(w), s)$ or $(w, t(w), (s, t(w), s', s''))$ is the priority $p(s)$. (Standard initial position: $(\epsilon, t(\epsilon), s_0)$)
Tree Automata and Games

Lemma

The tree automaton $A$ accepts an input tree $t$ iff in the parity game over $G_{A,t}$ there is a winning strategy for player Automaton from the initial position $(\epsilon, t(\epsilon), s_0)$.

Proof.

A successful run of $A$ on $t$ yields a winning strategy for Automaton in the parity game over $G_{A,t}$: Along each path the suitable choice of transitions is fixed by the run.

Conversely, a winning strategy for Automaton over $G_{A,t}$ clearly provides a method to build up a successful run of $A$ on $t$. Just apply this winning strategy along arbitrary paths.
Summary: Tree Automaton

- Tree Automata can be viewed as games between Automaton and Pathfinder
- Parity and Muller tree automata can be reduced to each other
- (Same holds for Rabin/Streett, Parity, and Muller tree automata)
- Radu showed closure properties of Muller tree automaton (union, intersection, projection)
- Missing: complementation
Complementation of Parity Tree Automaton

We will show basic idea.

- To complement a given automaton $A$ means to construct an automaton $B$ s.t.
  \[ t \notin A \leftrightarrow t \in B \]

- Due to the run lemma, complementation means to conclude from the non-existence of a winning strategy of Player Automaton in $G_{A,t}$ that there exists a winning strategy of Automaton in $G_{B,t}$.

Proof has two steps:

1. use determinacy of parity games to show that if Automaton has no winning strategy over $G_{A,t}$, then Pathfinder has a winning strategy over $G_{A,t}$ (from $(\epsilon, t(\epsilon), s_0)$)

2. Convert Pathfinder’s strategy into an Automaton strategy.
Complementation of Parity Tree Automaton

Theorem

For any parity tree automaton $A$ over $\Sigma$, one can construct a Muller tree automaton (and therefore a parity tree automaton) $B$ over $\Sigma$ that recognizes $T^\omega(\Sigma) \setminus \mathcal{L}(A)$

Proof.

From Step 1 (determinacy of parity games), we know there exists a (memoryless) winning strategy $f : S_1 \to \{0, 1\}$ for Player Pathfinder.

$$f : \{0, 1\}^* \times \Sigma \times T \to \{0, 1\}$$

decompose $f$ into a family of strategies parameterized by $w \in \{0, 1\}^*$

$$f_w : \Sigma \times T \to \{0, 1\}$$
Complementation of Parity Tree Automaton

Let $I$ be the set of all possible local instructions $i : \Sigma \times T \rightarrow \{0, 1\}$. Then, $f$ can be represented as $I$-labeled binary tree $s$ with $s(w) = f_w$.

Let $s \cdot t$ be the corresponding $(I \times \Sigma)$-labeled tree

$$s \cdot t(w) = (s(w), t(w)) \text{ for } w \in \{0, 1\}^*.$$

Since $f$ exists, we know there is an $I$-labeled tree $s$ s.t. for all sequences $\tau_0\tau_1\ldots$ of transitions chosen by Automaton and for all paths (in path for the unique) $\pi \in \{0, 1\}^*$, the generated state sequence violates the parity condition.

Intuitively, $f$ tells the “new” automaton for every tree $t \not\in \mathcal{L}(A)$ which path to track for a given transition sequences in order to reject/accept the tree $t$. 
Complementation of Parity Tree Automaton

So, we know:

1. There exists an $I$-labeled tree $s$ such that $s \cdot t$ satisfies

2. for all $\pi \in \{0, 1\}^\omega$

3. for all $\tau_0\tau_1 \cdots \in T^\omega$

4. if the sequence $s|_\pi$ of local
   instructions applied to the sequence of tree labels $t|_\pi$ and
   the sequence $\tau_0\tau_1 \cdots$ produces the path $\pi$, then the state
   sequence determined by $\tau_0\tau_1 \cdots$ violates the parity
   condition.
Complementation of Parity Tree Automaton

- Condition 4 is a property of $\omega$-words over $I \times \Sigma \times T \times \{0, 1\}$, which can be checked by a Muller word automaton $M_4$.

- Condition 3 is a property of $\omega$-words over $I \times \Sigma \times \{0, 1\}$ checked by $M_3$, which results from $M_4$ by universally quantifying $T$ (negate, project, negate).

- Condition 2 is a property of $(I \times \Sigma)$-labeled trees, which can be checked by a Muller tree automaton $M_2$ that simulates $M_3$ along each path.

- Condition 1, apply nondeterminism, a Muller tree automaton $B$ can be built by guessing a tree $s$ on the input tree $t$ and running $M_2$ on $s \cdot t$. 