Automata on Infinite Trees
Büchi Automata on Infinite Trees
**Definition**

A Büchi tree automaton over $\Sigma$ is $A = \langle S, I, T, F \rangle$, where:

- $S$ is a finite set of *states*,
- $I \subseteq S$ is a set of *initial states*,
- $T : S \times \Sigma \rightarrow 2^{S \times S}$ is the *transition relation*,
- $F \subseteq S$ is the set of *final states*. 
**Runs**

A *run* of $A$ over a tree $t : \{0, 1\}^* \rightarrow \Sigma$ is a mapping $\pi : \{0, 1\}^* \rightarrow S$ such that, for each position $p \in \{0, 1\}^*$, where $q = \pi(p)$, we have:

- if $p = \epsilon$ then $q \in I$, and
- if $q_i = \pi(pi)$, $i = 0, 1$ then $\langle q_0, q_1 \rangle \in T(q, t(p))$.

If $\pi$ is a *run* of $A$ and $\sigma$ is a *path* in $t$, let $\pi|_{\sigma}$ denote the path in $\pi$ corresponding to $\sigma$.

A run $\pi$ is said to be *accepting*, if and only if for every path $\sigma$ in $t$ we have:

$$\inf(\pi|_{\sigma}) \cap F \neq \emptyset$$
Closure Properties

For every Büchi automaton $A$ there exists a complete Büchi automaton $A'$ such that $\mathcal{L}(A) = \mathcal{L}(A')$.

**Theorem 1** The class of Büchi-recognizable tree languages is closed under union, intersection and projection.

Let $A_i = \langle S_i, I_i, T_i, F_i \rangle$, $i = 1, 2$, where $S_1 \cap S_2 = \emptyset$.

Let $A_\cup = \langle S_1 \cup S_2, I_1 \cup I_2, T_1 \cup T_2, F_1 \cup F_2 \rangle$. 
Closure Properties

Let $A_\cap = \langle S, I, T, F \rangle$ where:

- $S = S_1 \times S_2 \times \{0, 1, 2\}$
- $I = I_1 \times I_2 \times \{1\}$
- for any $s, s_1, s_2 \in S_1$, $s', s'_1, s'_2 \in S_2$, $a, b \in \{0, 1, 2\}$:
  $$\langle (s_1, s'_1, b), (s_2, s'_2, b) \rangle \in T((s, s', a), \sigma)$$
  iff $\langle s_1, s_2 \rangle \in T(s, \sigma)$, $\langle s'_1, s'_2 \rangle \in T(s', \sigma)$ and:
  
1. if $a = 0$ or ($a = 1$ and $s \notin F_1$), then $b = 1$
2. if ($a = 1$ and $s \in F_1$) or ($a = 2$ and $s \notin F_1$), then $b = 2$
3. if $a = 2$ and $s' \in F_2$, then $b = 0$
- $F = S \times S \times \{0\}$
Emptiness of Büchi Tree Automata

Let $A = \langle S, I, T, F \rangle$ be a Büchi tree automaton where $F = \{s_1, \ldots, s_m\}$, and $\pi : \{0, 1\}^* \rightarrow S$ be a successful run of $A$ on the tree $t \in \mathcal{T}(\Sigma)$.

For any $s \in S$, and any $u \in \{0, 1\}^*$ such that $\pi(u) = s$, let

$$d_\pi^u = \{ w \in u \cdot \{0, 1\}^* | \forall v. u < v < w \Rightarrow \pi(v) \notin F \}$$

By König’s lemma, $d_\pi^u$ is finite for any $u \in \{0, 1\}^*$.

Let $t_s^\pi$ be the restriction of $t$ to $d_\pi^u$. Let

$$T_s = \{ t_s^\pi | \pi \text{ is a successful run of } A \text{ on } t \}$$
Emptiness of Büchi Tree Automata

If \( \vec{s} = \langle s_1, \ldots, s_m \rangle \) are the final states of \( A \):

\[
\mathcal{L}(A) = \bigcup_{s_0 \in I} T_{s_0} \cdot \vec{s} \langle T_{s_1}, \ldots, T_{s_m} \rangle^{\omega \vec{s}}
\]

Conversely, the expression above denotes a Büchi-recognizable tree language.

Let \( A = \langle S, I, T, F \rangle \) be a Büchi tree automaton. For each \( s \in S \) let \( T_s \) be the recognizable tree language defined above. Eliminate from \( S \) (and \( T \)) all states \( s \) such that \( T_s = \emptyset \), and let \( S' \) be the resulting set of states.

We claim that \( \mathcal{L}(A) \neq \emptyset \iff S' \cap I \neq \emptyset \).
The Complement Problem

Let $\Sigma = \{a, b\}$, $T_0 = \{t \in T^\omega(\Sigma) \mid \text{some path in } t \text{ has infinitely many } a\text{'s}\}$

$T_0$ is Büchi recognizable.

Let $A = \langle \{s_0, s_1, s_a, s_b\}, \{s_0\}, T, \{s_1, s_a\} \rangle$, where $T$ is defined by:

\[
\begin{align*}
a(s_0, a, b) & \rightarrow \{\langle s_1, s_a \rangle, \langle s_a, s_1 \rangle\} \\
b(s_0, a, b) & \rightarrow \{\langle s_1, s_b \rangle, \langle s_b, s_1 \rangle\} \\
a(s_1) & \rightarrow \{\langle s_1, s_1 \rangle\} \\
b(s_1) & \rightarrow \{\langle s_1, s_1 \rangle\}
\end{align*}
\]
The Complement Problem

Let $\mathcal{T}_1 = \mathcal{T}^\omega(\Sigma) \setminus \mathcal{T}_0 = \{t \in \mathcal{T}^\omega(\Sigma) \mid \text{all paths in } t \text{ have finitely many } a \text{'s}\}$. We show that $\mathcal{T}_1$ cannot be recognized by a Büchi tree automaton.

Exercise 1 $I = \{s_0, s_1\}$, $F = \{s_1\}$ and

\[
\begin{align*}
a(s_0) & \rightarrow \langle s_0, s_0 \rangle \\
& \quad \langle s_0, s_1 \rangle \\
& \quad \langle s_1, s_0 \rangle \\
& \quad \langle s_1, s_1 \rangle \\

b(s_0) & \rightarrow \langle s_0, s_0 \rangle \\
& \quad \langle s_0, s_1 \rangle \\
& \quad \langle s_1, s_0 \rangle \\
& \quad \langle s_1, s_1 \rangle \\

b(s_1) & \rightarrow \langle s_1, s_1 \rangle 
\end{align*}
\]
The Complement Problem

Let $T_n : \{0, 1\}^* \to \Sigma$ be the language of trees:

$$t_n(p) = \begin{cases} 
    a & \text{if } p \in \{\varepsilon, 1^{m_1}0, 1^{m_1}01^{m_2}0, \ldots, 1^{m_1}01^{m_2}0\ldots 1^{m_n}0 \mid m_1, \ldots m_n \in \mathbb{N}\} \\
    b & \text{otherwise}
\end{cases}$$

Obviously, $T_n \subset \mathcal{T}_1$, for all $n \in \mathbb{N}$.

Suppose there exists a Büchi automaton $A = \langle S, I, T, F \rangle$ with $k$ states, s.t. $\mathcal{L}(A) = \mathcal{T}_1$. Let $\pi$ be the accepting run of $A$ over $t_{k+1}$. Then there exist:

- $m_1 > 0$ such that $\pi(1^{m_1}) = s_1 \in F$
- $m_2 > 0$ such that $\pi(1^{m_1}01^{m_2}) = s_2 \in F$
- $\ldots$

There exists a path $\sigma$ in $t_m$ and $u < v < w < \sigma$, such that $\pi(u) = \pi(w) = s \in F$ and $t_m(v) = a$. Then $\pi = r_1 \cdot s r_2 \cdot s r_3$, and $r_1 \cdot s r_2^{\omega} s$ is a successful run on $q_1 \cdot q_2^{\omega}$, which contains a path with infinitely many $a$. 
Muller Automata on Infinite Trees
**Definition**

A Muller tree automaton $\Sigma$ is $A = \langle S, I, T, F \rangle$, where:

- $S$ is a finite set of *states*,
- $I \subseteq S$ is a set of *initial states*,
- $T : S \times \Sigma \rightarrow 2^{S \times S}$ is the *transition function*,
- $F \subseteq 2^S$, is the set of *accepting sets*.

A run $\pi$ of $A$ over $t$ is said to be *accepting*, iff for every path $\sigma$ in $t$:

$$\inf(\pi|_\sigma) \in F$$
Closure Properties

The class of Muller-recognizable tree languages is closed under union and intersection.

For **union**, the proof is exactly as in the case of Büchi automata. For $A_\cup$, the set of accepting sets is the union of the sets $\mathcal{F}_i, i = 1, 2$.

For **intersection**, let $A_\cap = \langle S_1 \times S_2, I_1 \times I_2, T, \mathcal{F} \rangle$, where:

- $\langle (s_1, s'_1), (s_2, s'_2) \rangle \in T((s, s'), \sigma)$ iff $\langle s_1, s_2 \rangle \in T(s, \sigma)$ and $\langle s'_1, s'_2 \rangle \in T(s', \sigma)$, and

- $\mathcal{F} = \{ G \in S_1 \times S_2 \mid pr_1(G) \in \mathcal{F}_1$ and $pr_2(G) \in \mathcal{F}_2 \}$, where:
  - $pr_1(G) = \{ s \in S_1 \mid \exists s'. (s, s') \in G \}$, and
  - $pr_2(G) = \{ s \in S_2 \mid \exists s'. (s', s) \in G \}$. 
Rabin Automata on Infinite Trees
**Definition**

A **Rabin** tree automaton $\Sigma$ is $A = \langle S, I, T, \Omega \rangle$, where:

- $S$ is a finite set of *states*,
- $I \subseteq S$ is a set of *initial states*,
- $T : S \times \Sigma \to 2^{S\times S}$ is the *transition function*,
- $\Omega = \{\langle N_1, P_1 \rangle, \ldots, \langle P_n, N_n \rangle\}$ is the set of *accepting pairs*.

A run $\pi$ of $A$ over $t$ is said to be *accepting*, if and only if for every path $\sigma$ in $t$ there exists a pair $\langle N_i, P_i \rangle \in \Omega$ such that:

$$\inf(\pi|_\sigma) \cap N_i = \emptyset \text{ and } \inf(\pi|_\sigma) \cap P_i \neq \emptyset$$
Büchi, Muller and Rabin

For every Büchi tree automaton $A$ there exists a Muller tree automaton $B$, such that $\mathcal{L}(A) = \mathcal{L}(B)$, but not viceversa.

For every Muller tree automaton $A$ there exists a Rabin tree automaton $B$, such that $\mathcal{L}(A) = \mathcal{L}(B)$, and viceversa.
From Büchi to Muller

For each (nondeterministic) Büchi automaton $A$ there exists a (nondeterministic) Muller automaton $B$ such that $\mathcal{L}(A) = \mathcal{L}(B)$

Let $A = \langle S, I, T, F \rangle$ be a Büchi automaton.

Define $B = \langle S, I, T, \{ G \in 2^S \mid G \cap F \neq \emptyset \} \rangle$

Allowing Muller automata to be nondeterministic is essential here.
From Rabin to Muller

Given a Rabin automaton $A = \langle S, I, T, \Omega \rangle$, such that

$$\Omega = \{\langle N_1, P_1 \rangle, \ldots, \langle N_k, P_k \rangle\}$$

let $B = \langle S, I, T, \mathcal{F} \rangle$ be the Muller automaton, where

$$\mathcal{F} = \{F \subseteq S \mid F \cap N_i = \emptyset \text{ and } F \cap P_i \neq \emptyset \text{ for some } 1 \leq i \leq k\}$$
From Muller to Rabin

Given a Muller automaton \( A = \langle S, I, T, \mathcal{F} \rangle \), there exists a Rabin automaton \( B \) such that \( \mathcal{L}(A) = \mathcal{L}(B) \)

Let \( \mathcal{F} = \{Q_1, \ldots, Q_k\} \)

Let \( B = \langle S', I', T', \Omega' \rangle \) where:

- \( S' = 2^{Q_1} \times \ldots \times 2^{Q_k} \times S \)
- \( I' = \{\langle \emptyset, \ldots, \emptyset, s_0 \rangle \mid s_0 \in I\} \)
From Muller to Rabin

- $T'(\langle S_1, \ldots, S_k, s \rangle, a) = (\langle S'_1, \ldots, S'_k, s' \rangle, \langle S''_1, \ldots, S''_k, s'' \rangle)$ where:
  - $T(s, a) = (s', s'')$
  - for all $1 \leq i \leq k$:
    
    $S'_i = S''_i = \begin{cases} 
    \emptyset, & \text{if } S_i \cup \{s\} = Q_i \\
    (S_i \cup \{s\}) \cap Q_i, & \text{otherwise}
    \end{cases}$

- $P_i = \{\langle S_1, \ldots, S_i, \ldots, S_k, s \rangle \mid S_i = Q_i\}, 1 \leq i \leq k$

- $N_i = \{\langle S_1, \ldots, S_i, \ldots, S_k, s \rangle \mid s \not\in Q_i\}, 1 \leq i \leq k$
The Rabin Complementation Theorem

**Theorem 2 (Rabin ’69)** The class of Rabin-recognizable tree languages is closed under complement.

The class of Rabin-recognizable tree languages is closed under union and intersection, because Muller-recognizable languages are.
Emptiness of Rabin Automata

Given an alphabet \( \Sigma \), an infinite tree \( t \in T^\omega(\Sigma) \) is said to be \textit{regular} if there are only finitely many distinct subtrees \( t_u \) of \( t \), where \( u \in \{0, 1\}^* \).

\textbf{Example 1} The infinite binary tree \( f(g(f(\ldots), f(\ldots)), g(f(\ldots), f(\ldots))) \) is regular. \qed

\textbf{Theorem 3 (Rabin ’72)}

1. Any non-empty Rabin-recognizable set of trees contains a regular tree.
2. The emptiness problem for Rabin tree automata is decidable.
Reduction to empty alphabet

Let $A = \langle S, I, T, \Omega \rangle$ be a Rabin tree automaton over $\Sigma$, such that $L(A) \neq \emptyset$, where $\Omega = \{\langle N_1, P_1 \rangle, \ldots, \langle N_n, P_n \rangle\}$.

Let $A' = \langle S \times \Sigma, I \times \Sigma, T', \Omega' \rangle$, where:

- $\langle (s_1, \sigma_1), (s_2, \sigma_2) \rangle \in T'((s, \sigma))$ iff $\langle s_1, s_2 \rangle \in T(s, \sigma)$, and $\sigma_1, \sigma_2 \in \Sigma$.
- $\Omega' = \{\langle N_1 \times \Sigma, P_1 \times \Sigma \rangle, \ldots, \langle N_n \times \Sigma, P_n \times \Sigma \rangle\}$.

The successful runs of $A'$ are pairs $(\pi, t)$, where $t \in L(A)$, and $\pi$ is a successful run of $A$ on $t$. 
**Regular successful runs**

For any Rabin tree automaton $A$, there exists a Rabin tree automaton $A'$ with one initial state such that $\mathcal{L}(A) = \mathcal{L}(A')$.

Consider a Rabin tree automaton $A = \langle S, s_0, T, \Omega \rangle$ over the empty alphabet, and let $\pi$ be a successful run of $A$.

**Claim 1** If $A$ has a successful run, $A$ has also a regular successful run.

A state $s \in S$ is said to be *live* if $s \neq s_0$ and $\langle s_1, s_2 \rangle \in T(s)$ for some $s_1, s_2 \in S$, where either $s_1 \neq s$ or $s_2 \neq s$.

By induction on $n = \text{the number of live states in } A$. 
Regular successful runs

Base case $n = 0$: $\pi(\epsilon) = s_0$ and $\pi(p) = s$, for all $p \in dom(\pi)$, and $s \in S$ is non-live.

Inductive step $n > 0$:

Case 1 If some live state in $A$ is missing on $\pi$, apply the induction hypothesis.

Case 2 All live states of $A$ appear on $\pi$, and there is a position $u \in \{0, 1\}^*$ such that $\pi(u) = s$ is live, but some live state $s'$ does not appear in $\pi_u$.

Let $\pi_1 = \pi \setminus \pi_u$ and $\pi_2 = \pi_u$. Both $\pi_1$ and $\pi_2$ are runs of automata with $n - 1$ live states, hence there exists successful regular runs $\pi'_1$ and $\pi'_2$ of these automata. The desired run is $\pi'_1 \cdot s \pi'_2$. 
Regular successful runs

**Case 3** All live states appear in any subtree of $\pi$. Let $\sigma$ be a path in $\pi$ consisting of all the live states appearing again and again, and only of the live states, with the exception of $\pi(\epsilon)$. Q: Why does $\sigma$ exist?

There exists $\langle N, P \rangle \in \Omega$, such that $\inf(\sigma) \cap N = \emptyset$ and $\inf(\sigma) \cap P \neq \emptyset$. Then $N$ contains only non-live states.

Let $s \in \inf(\sigma) \cap P$ and $u, v$ be the 1$^{st}$ and 2$^{nd}$ positions such that $\sigma(u) = \sigma(v) = s$.

Let $\pi_1 = \pi \setminus \pi_u$ and $\pi_2 = \pi_u \setminus \pi_v$. Both $\pi_1$ and $\pi_2$ are runs of automata with $n - 1$ live states, hence there exists successful regular runs $\pi'_1$ and $\pi'_2$ of these automata. The desired run is $\pi'_1 \cdot_s \pi'_2 \omega^s$. 
The Emptiness Problem

Let $A$ be an input-free Rabin tree automaton with $n$ live states.

We derive $A_{n-1}, A_{n-2}, \ldots, A_0$ from $A$, having $n - 1, n - 2, \ldots 0$ live states.

If $A$ has a successful run, then it it has a regular run, composed of runs of $A_{n-1}, A_{n-2}, \ldots, A_0$.

So it is enough to check emptiness of $A_{n-1}, A_{n-2}, \ldots, A_0$. 
Rabin Automata, SkS and SωS
Defining infinite paths

We say that a set of positions \( X \) is linear iff the following holds:

\[
\text{linear}(X) : (\forall x, y . X(x) \land X(y) \rightarrow x \leq y \lor y \leq x)
\]

\( X \) is a path iff:

\[
\text{path}(X) : \text{linear}(X) \land \forall Y . \text{linear}(Y) \land X \subseteq Y \rightarrow X = Y
\]
From Automata to Formulae

Let $A = \langle S, I, T, \Omega \rangle$ be a Rabin tree automaton, where $S = \{s_1, \ldots, s_p\}$.

Let $\vec{Y} = \{Y_1, \ldots, Y_p\}$ be set variables.

If $X$ denotes a path, state $i$ appears infinitely often in $X$ iff:

$$\inf_{i}(X) : \forall x . X(x) \rightarrow \exists y . x \leq y \land X(y) \land Y_i(y)$$

The formula expressing the accepting condition is:

$$\Phi_\Omega(\vec{Y}) : \forall X . path(X) \rightarrow \bigvee_{\langle N,P \rangle \in \Omega} \left( \bigwedge_{s_i \in N} \neg \inf_{i}(X) \land \bigvee_{s_i \in P} \inf_{i}(X) \right)$$
Decidability of S2S

**Theorem 4** Given an alphabet $\Sigma$, a tree language $L \subseteq T^\omega(\Sigma)$ is definable in S2S iff it is recognizable.

**Corollary 1** The SAT problem for S2S is decidable.
Obtaining Decidability Results by Reduction

Suppose we have a logic $L$ interpreted over the domain $D$, such that the following problem is decidable:

for each formula $\varphi$ of $L$ there exists $m \in D$ such that $m \models \varphi$

Then we can prove the same thing for another logic $L'$ interpreted over $D'$ iff there exists functions $\Delta : D' \to D$ and $\Lambda : L' \to L$ such that for all $m' \in D'$ and $\varphi' \in L$ we have:

$m' \models \varphi' \iff \Delta(m') \models \Lambda(\varphi')$
Decidability of $S^\omega S$

Every tree $t : \mathbb{N}^* \rightarrow \Sigma$ can be encoded as $t' : \{0, 1\}^* \rightarrow \Sigma$.

Let $D = \{\epsilon\} \cup \{1^{n_1+1}01^{n_2+1}0\ldots1^{n_k+1}0 \mid k \geq 1, \ n_i \in \mathbb{N}, \ 1 \leq i \leq k\}$.

Embedding the domain of $S^\omega S$ into $S2S$:

$$D(x) : \exists z \forall y . \ z \leq y \land x = z \lor \forall y . \ s_0(y) \leq x \rightarrow \exists y' . \ y = s_1(y')$$
Decidability of $S\omega S$

If $p = 1^{n_1+1}01^{n_2+1}0\ldots 1^{n_k+1}0$, let

$$f_i(p) = p \cdot 1^{i+1}0 = 1^{n_1+1}01^{n_2+1}0\ldots 1^{n_k+1}01^{i+1}0$$

$x \preceq_D y : D(x) \land D(y) \land x \leq y$

Define the relation $x \preceq_D y$ iff $x \in D$ and $y = x \cdot 1^{n+1}0$, for some $n \in \mathbb{N}$:

$x \preceq_D y : \exists z \cdot y = s_0(z) \land \forall z' . x \leq z \land z' < z \rightarrow s_1(z') \leq y$

Define $f_0, f_1, f_2, \ldots$ by induction:

$$f_0(x) = y : D(x) \land D(y) \land y = s_0(x)$$

$$f_{i+1}(x) = y : D(x) \land D(y) \land x \preceq_D y \land \forall z . \ x \preceq_D z \land$$

$$\land_{0 \leq k \leq i} z \neq f_k(x) \rightarrow y \preceq_D z$$