Reachability and Büchi Games

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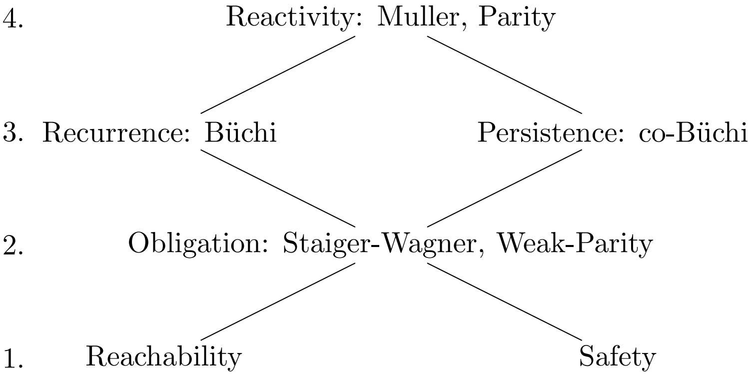
Grenoble, December 2018

Hierarchy


4. Reactivity: Muller, Parity
3. Recurrence: Büchi
Persistence: co-Büchi
2. Obligation: Staiger-Wagner, Weak-Parity
1. Reachability
Safety

Reachability and Safety Games

Reachability and Safety Games

Theorem

Given a reachability game (G, F) with $G = (S, S_0, E)$ and $F \subseteq S$, then the winning regions W_0 and W_1 of Player 0 and 1, respectively, are computable, and both players have corresponding memoryless winning strategies.

Proof.

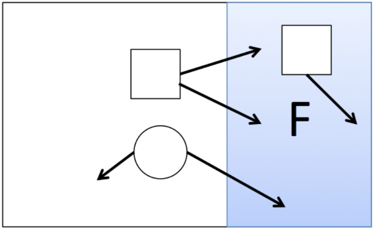
Define

$$\mathrm{Attr}_0^i(F) := \{s \in S \mid \text{Player 0 can force a visit from } s \text{ to } F \text{ in less than } i \text{ moves}\}$$

Force Visit in Next Step

Given a set of states, compute the set of states $\text{ForceNext}_0(F)$ from which of Player 0 can force to visit F in the next step. I.e., for each state $s \in \text{ForceNext}_0(F)$ Player 0 can fix a strategy s.t. all plays starting in s visit F in the first step.

$$\begin{aligned}\text{ForceNext}_0(F) &= \{s \in S_0 \mid \exists s' \in S : (s, s') \in E \wedge s' \in F\} \cup \\ &\quad \{s \in S_1 \mid \forall s' \in S : (s, s') \in E \rightarrow s' \in F\}\end{aligned}$$

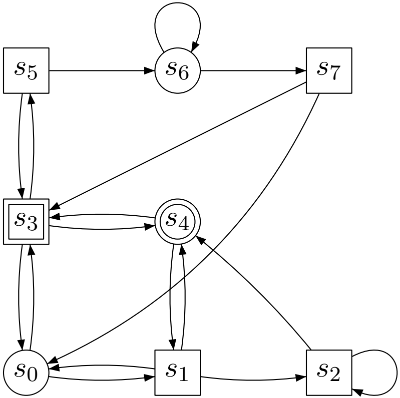


Computing the Attractor

Construction of $\mathrm{Attr}_0^i(F)$:

$$\begin{aligned}\mathrm{Attr}_0^0(F) &= F \\ \mathrm{Attr}_0^{i+1}(F) &= \mathrm{Attr}_0^i(F) \cup \mathrm{ForceNext}_0(\mathrm{Attr}_0^i(F))\end{aligned}$$

Example



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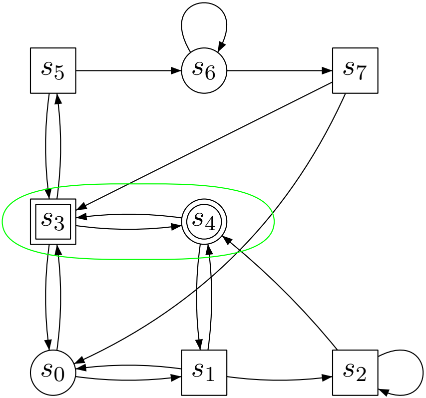

$$\mathrm{Attr}_0^0 = \{s_3, s_4\}$$

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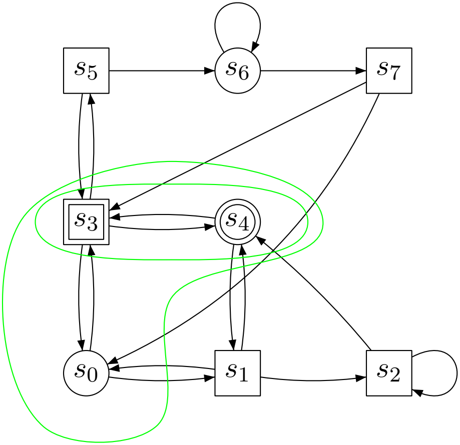

$\mathrm{Attr}_0^0 = \{s_3, s_4\}$
$\mathrm{Attr}_0^1 = \{s_0, s_3, s_4\}$

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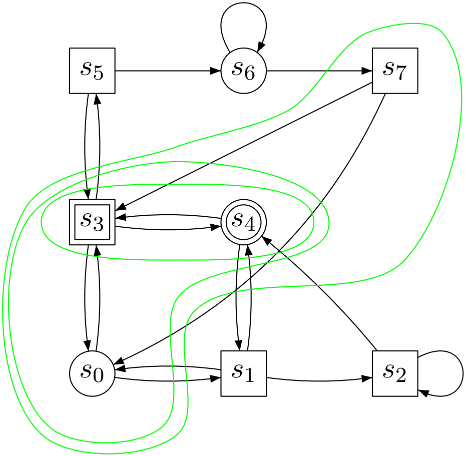

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$$\mathrm{Attr}_0^2 = \{s_0, s_3, s_4, s_7\}$$

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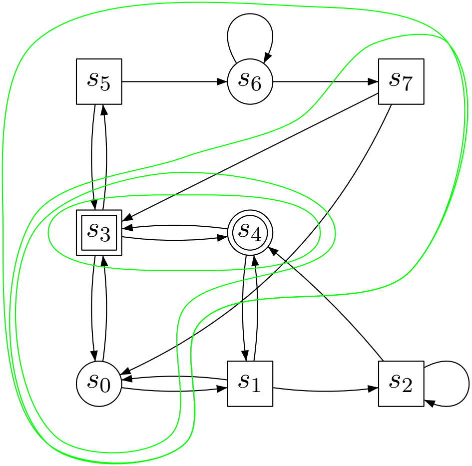

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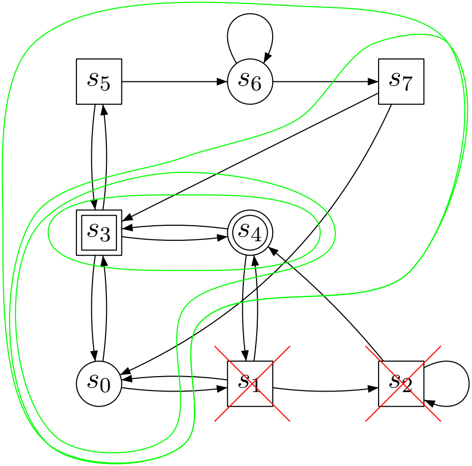

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Computing the Attractor

Construction of $\mathrm{Attr}_0^i(F)$:

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Then $\mathrm{Attr}_0^0(F) \subseteq \mathrm{Attr}_0^1(F) \subseteq \mathrm{Attr}_0^2(F) \subseteq \dots$ and since S is finite, there exists $k \leq |S|$ s.t. $\mathrm{Attr}_0^k(F) = \mathrm{Attr}_0^{k+1}(F)$.

The 0-Attractor is defined as:

$$\mathrm{Attr}_0(F) := \bigcup_{i=0}^{|S|} \mathrm{Attr}_0^i(F)$$

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$$\mathrm{Attr}_0(F) := \bigcup_{i=0}^{|S|} \mathrm{Attr}_0^i(F)$$

Claim: $W_0 = \mathrm{Attr}_0(F)$ and $W_1 = S \setminus \mathrm{Attr}_0(F)$

Duality Between Players

Assume we have a partition of the state space $S = P_0 \cup P_1$ (i.e., $P_0 \cap P_1 = \emptyset$) and we want to prove $W_0 = P_0$ and $W_1 = P_1$.

We want to prove $P_0 \supseteq W_0$, $P_0 \subseteq W_0$, $P_1 \supseteq W_1$, and $P_1 \subseteq W_1$.

Since we know that $W_0 \cap W_1 = \emptyset$ holds, it is sufficient to prove $P_0 \subseteq W_0$ and $P_1 \subseteq W_1$.

$$\begin{aligned} P_0 &\subseteq W_0 \\ S \setminus P_0 &\supseteq S \setminus W_0 \\ P_1 &\supseteq S \setminus W_0 \supseteq W_1 \\ P_1 &\supseteq W_1 \end{aligned}$$

$$\begin{aligned} P_1 &\subseteq W_1 \\ S \setminus P_1 &\supseteq S \setminus W_1 \\ P_0 &\supseteq S \setminus W_1 \supseteq W_0 \\ P_0 &\supseteq W_0 \end{aligned}$$

0-Attractor

To show $W_0 = \text{Attr}_0(F)$ and $W_1 = S \setminus \text{Attr}_0(F)$, we construct winning strategies for Player 0 and 1.

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Proof.

$\text{Attr}_0(F) \subseteq W_0$

We prove for every i and for every state $s \in \text{Attr}_0^i(F)$ that Player 0 has a positional winning strategy to reach F in $\leq i$ steps.

- (Base) $s \in \text{Attr}_0^0(F) = F$
- (Induction) $s \in \text{Attr}_0^{i+1}(F)$

  If $s \in \text{Attr}_0^i(F)$, then we apply induction hypothesis.

  Otherwise $s \in \text{ForceNext}_0(\text{Attr}_0^i(F)) \setminus \text{Attr}_0^i(F)$ and Player 0 can force a visit to $\text{Attr}_0^i(F)$ in one step and from there she needs at move i steps by induction hypothesis. So, F is reached after a finite number of moves.

0-Attractor cont.

Proof cont.

$S \setminus \mathrm{Attr}_0(F) \subseteq W_1$

Assume $s \in S \setminus \mathrm{Attr}_0(F)$, then $s \notin \mathrm{ForceNext}_0(\mathrm{Attr}_0(F))$ and we have two cases:

(a) $s \in S_0 \cap S \setminus \mathrm{Attr}_0(F)\ \forall s' \in S\colon (s, s') \in E \rightarrow s' \notin \mathrm{Attr}_0(F)$

(b) $s \in S_1 \cap S \setminus \mathrm{Attr}_0(F)\ \exists s' \in S\colon (s, s') \in E \wedge s' \notin \mathrm{Attr}_0(F)$

In $S \setminus \mathrm{Attr}_0(F)$ Player 1 can choose edges according to (b) leading again to $S \setminus \mathrm{Attr}_0(F)$ and by (a) Player 0 cannot escape from $S \setminus \mathrm{Attr}_0(F)$. So, F will be avoided forever.

$$W_0 = \mathrm{Attr}_0(F) \text{ and } W_1 = S \setminus \mathrm{Attr}_0(F)$$

Safety Games

Given a safety game (G, F) with $G = (S, S_0, E)$, i.e.,

$$\phi_S = \{\rho \in S^\omega \mid \mathrm{Occ}(\rho) \subseteq F\},$$

consider the reachability game $(G, S \setminus F)$, i.e.,

$$\phi_R = \{\rho \in S^\omega \mid \mathrm{Occ}(\rho) \cap (S \setminus F) \neq \emptyset\}.$$

$$\begin{aligned}\text{Then, } S^\omega \setminus \phi_R &= \{\rho \in S^\omega \mid \mathrm{Occ}(\rho) \cap (S \setminus F) = \emptyset\} \\ &= \{\rho \in S^\omega \mid \mathrm{Occ}(\rho) \subseteq F\}.\end{aligned}$$

Player 0 has a safety objective in (G, F).

Player 1 has a reachability objective in (G, F).

So, W_0 in the safety game (G, F) corresponds to W_1 in the reachability game $(G, S \setminus F)$.

Summary

We know how to solve reachability and safety games by positional winning strategies.

The strategies are

- Player 0: Decrease distance to F
- Player 1: Stay outside of $\mathrm{Attr}_0(F)$

In LTL, $\Diamond F$ = reachability and $\Box F$ = safety.

Next, $\Box\Diamond F$ = Büchi and $\Diamond\Box F$ = Co-Büchi.

Exercise

1. Given a reachability game (G, F) with $G = (S, S_0, E)$ and $F \subseteq Q$, give an algorithm that computes the 0-Attractor(F) in time $O(|E|)$.

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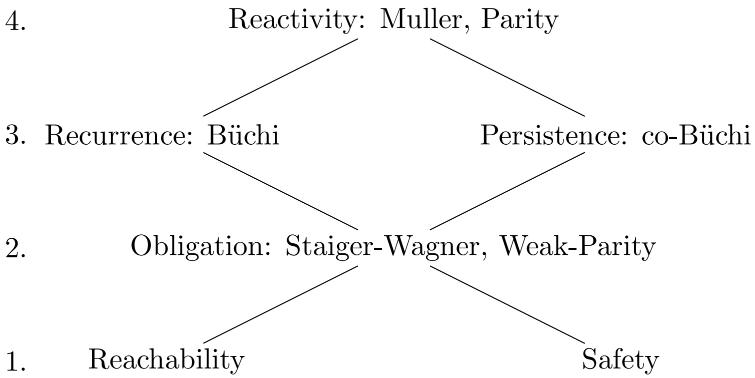
Solution:

1. Preprocessing: Compute for every state $s \in S_1$ outdegree $\mathrm{out}(s)$
2. Set $n(s) := \mathrm{out}(s)$ for each $s \in S_1$
3. To breadth-first search backwards from F with the following conventions:
   - mark all $s \in F$
   - mark $s \in S_0$ if reached from marked state
   - mark $s \in S_1$ if $n(s) = 0$, other set $n(s) := n(s) - 1$.

The marked vertices are the ones of $\mathrm{Attr}_0(F)$.

Hierarchy



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Büchi and co-Büchi Games

Büchi Game

Given a Büchi game (G, F) over the game graph $G = (S, S_0, E)$ with the set $F \subseteq S$ of Büchi states, we aim to

- determine the winning regions of Player 0 and 1
- compute their respective winning strategies

Recall, Player 0 wins ρ iff she visits infinitely often states in F, i.e., $\phi = \{\rho \in S^\omega \mid \inf(\rho) \cap F \neq \emptyset\}$.

Idea

Compute for $i \geq 1$ the set Recur_0^i of accepting states $s \in F$ from which Player 0 can force at least i revisits to F.

Then, we will show that

$$F \supseteq \mathrm{Recur}_0^1(F) \supseteq \mathrm{Recur}_0^2(F) \supseteq \dots$$

and we compute the winning region of Player 0 with

$$\mathrm{Recur}_0(F) := \bigcap_{i \leq 1} Recur_0^i(F)$$

Again, since F is finite, there exists k such that $\mathrm{Recur}_0(F) = \mathrm{Recur}_0^k(F)$.

Claim: $W_0 = \mathrm{Attr}_0(Recur_0(F))$

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First, we define Recur_0 formally using a modified version of Attractor.

One-Step Attractor

We count revisits, so we need the set of states from which Player 0 can force a revisit to F, i.e., state from which she can force a visit in ≥ 1 steps.

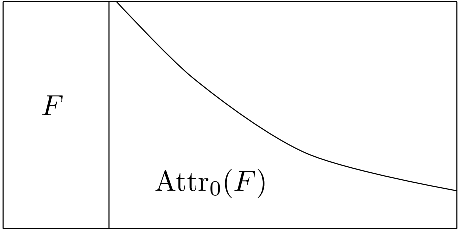
We define a slightly modified attractor:

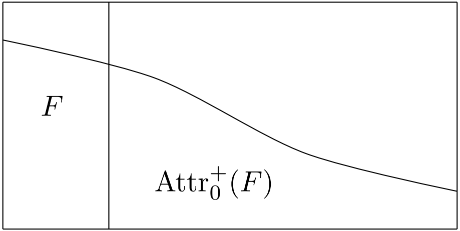
$$\begin{aligned} A_0^0 &= \emptyset \\ A_0^{i+1} &= A_0^i \cup \text{ForceNext}(A_0^i \cup F) \end{aligned}$$

$$\text{Attr}_0^+(F) = \bigcup_{i \geq 0} A_0^i$$

$\text{Attr}_0^+(F)$ is the set of states from which Player 0 can force a revisit to F.

Visit versus Revisit





Recurrence Set

We define

$$\mathrm{Recur}_0^0(F) := F$$
$$\mathrm{Recur}_0^{i+1}(F) := F \cap \mathrm{Attr}_0^+(\mathrm{Recur}_0^i(F))$$
$$\mathrm{Recur}_0(F) := \bigcap_{i \geq 0} \mathrm{Recur}_0^i(F)$$

We show that there exists k such that $\mathrm{Recur}_0(F) := \bigcap_{i \geq 0}^{k} \mathrm{Recur}_0^i(F)$ by proving $\mathrm{Recur}_0^{i+1}(F) \subseteq \mathrm{Recur}_0^i(F)$ for all $i \geq 0$.

Proof.

- $i = 0$: $F \cap \mathrm{Attr}_0^+(F) \subseteq F$
- $i \to i+1$:
  $\mathrm{Recur}_0^{i+1}(F) = F \cap \mathrm{Attr}_0^+(\mathrm{Recur}_0^i(F)) \subseteq F \cap \mathrm{Attr}_0^+(\mathrm{Recur}_0^{i-1}(F))$ $= \mathrm{Recur}_0^i(F)$ since (i) $\mathrm{Recur}_0^i(F) \subseteq \mathrm{Recur}_0^{i-1}(F)$ by ind. hyp. and (ii) Attr_0^+ is monotone.

Recurrence Set cont.

We show that all states in $\mathrm{Attr}_0(\mathrm{Recur}_0(F))$ are winning for Player 0, i.e., $\mathrm{Attr}_0(\mathrm{Recur}_0(F)) \subseteq W_0$. We construct a memoryless winning strategy for Player 0 for all states in $\mathrm{Attr}_0(\mathrm{Recur}_0(F))$.

Proof.

We know that there exists k such that
$\mathrm{Recur}_0^{k+1}(F) = \mathrm{Recur}_0^k(F) = F \cap \mathrm{Attr}_0^+(\mathrm{Recur}_0^k(F))$. So,

- for $s \in \mathrm{Recur}_0^k(F) \cap S_0$ Player 0 can choose an edges back to $\mathrm{Attr}_0^+(\mathrm{Recur}_0^k(F))$ and
- for $s \in \mathrm{Recur}_0^k(F) \cap S_1$ all edges lead back to $\mathrm{Attr}_0^+(\mathrm{Recur}_0^k(F))$.

For all states in $\mathrm{Attr}_0(\mathrm{Recur}_0(F)) \setminus \mathrm{Recur}_0(F)$, Player 0 can follow the attractor strategy to reach $\mathrm{Recur}_0(F)$.

□

Recurrence Set cont.

We show $S \setminus \mathrm{Attr}_0(\mathrm{Recur}_0(F)) \subseteq W_1$.

Proof.

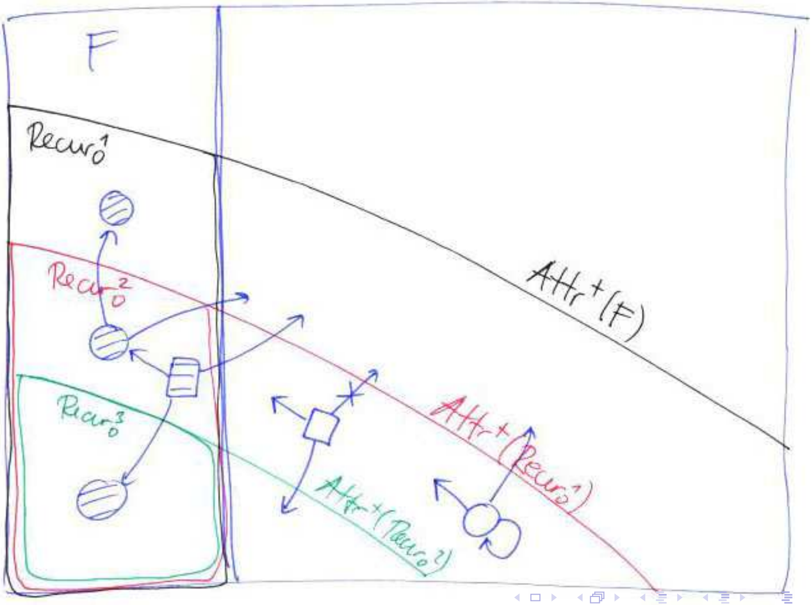
Show: Player 1 can force $\leq i$ visits to F from $s \notin \mathrm{Attr}_0(\mathrm{Recur}_0^i(F))$

$i = 0$: $s \notin \mathrm{Attr}_0(F)$, so Player 1 can avoid visiting F at all.

$i \to i+1$: $s \notin \mathrm{Attr}_0(\mathrm{Recur}_0^{i+1}(F))$.

- $s \notin \mathrm{Attr}_0(\mathrm{Recur}_0^i(F))$, Player 1 plays according to ind. hypothese
- Otherwise, $s \in \mathrm{Attr}_0(\mathrm{Recur}_0^i(F)) \setminus \mathrm{Attr}_0(\mathrm{Recur}_0^{i+1}(F))$ and Player 1 can avoid $\mathrm{Attr}_0(\mathrm{Recur}_0^{i+1}(F))$. In particular, $s \notin \mathrm{Recur}_0^{i+1}(F) = F \cap \mathrm{Attr}_0^+(\mathrm{Recur}_0^i(F))$.
  - If $s \in \mathrm{Recur}_0^i$, then Player 1 can force to leave $\mathrm{Attr}_0^+(\mathrm{Recur}_0^i(F))$, otherwise $s \in \mathrm{Recur}_0^{i+1}(F)$. (So, by ind. hyp. at most $i+1$ visits.)
  - If $s \in \mathrm{Attr}_0(\mathrm{Recur}_0^i(F)) \setminus \mathrm{Recur}_0^i(F)$, avoid $\mathrm{Attr}_0(\mathrm{Recur}_0^{i+1}(F))$.

Recurrence Set cont.



Büchi games

We have shown that Player 0 has a (memoryless) winning strategy from every state in $\mathrm{Attr}_0(\mathrm{Recur}_0(F))$, so $\mathrm{Attr}_0(\mathrm{Recur}_0(F)) \subseteq W_0$. And, Player 1 has a (memoryless) winning strategy from every state in $S \setminus \mathrm{Attr}_0(\mathrm{Recur}_0(F))$, so $S \setminus \mathrm{Attr}_0(\mathrm{Recur}_0(F)) \subseteq W_1$. This implies the following theorem.

Theorem

Given a Büchi game $((S, S_0, E), F)$, the winning regions W_0 and W_1 are computable and form a partition, i.e., $W_0 \cup W_1 = S$. Both players have memoryless winning strategies.

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How expensive it the computation of W_0 and W_1?

Co-Büchi Games

Given a Co-Büchi Game $((S, S_0, E), F)$, i.e.,

$$\phi_C = \{\rho \in S^\omega \mid \text{Inf}(\rho) \subseteq F\}$$

consider the Büchi Game $((S, S_0, E), S \setminus F)$, i.e,

$$\phi_B = \{\rho \in S^\omega \mid \text{Inf}(\rho) \cap S \setminus F \neq \emptyset\}.$$

$$\begin{aligned} \text{Then, } S^\omega \setminus \phi_B &= \{\rho \in S^\omega \mid \text{Inf}(\rho) \cap (S \setminus F) = \emptyset\} \\ &= \{\rho \in S^\omega \mid \text{Inf}(\rho) \subseteq F\}. \end{aligned}$$

Player 0 has a co-Büchi objective in (G, F).

Player 1 has a Büchi objective in (G, F).

So, W_0 in the co-Büchi game (G, F) corresponds to W_1 in the Büchi game $(G, S \setminus F)$.

Summary

We know how to solve Büchi and Co-Büchi games by positional winning strategies.

In LTL,

- $\Diamond F$ = reachability
- $\Box F$ = safety
- $\Box\Diamond F$ = Büchi
- $\Diamond\Box F$ = Co-Büchi

Exercise

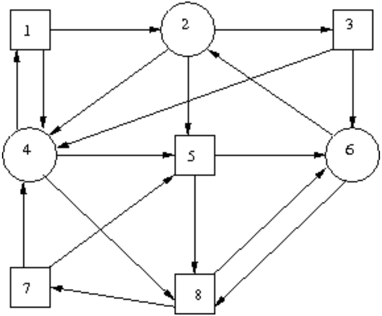
2. Consider the game graph shown in below and the following winning conditions:

   (a) $\mathrm{Occ}(\rho) \cap \{1\} \neq \emptyset$ and

   (b) $\mathrm{Occ}(\rho) \subseteq \{1, 2, 3, 4, 5, 6\}$ and

   (c) $\mathrm{Inf}(\rho) \cap \{4, 5\} \neq \emptyset$.

Compute the winning regions and corresponding winning strategies showing the intermediate steps (i.e., the Attractor and Recurrence sets) of the computation.



Exercise

3. Given a game graph $G = (S, S_0, T)$ and a set $F \subseteq S$. Let W_0 and W_1 be the winning regions of Player 0 and Player 1, respectively, in the Buchi game (G, F). Prove or disprove:
   (a) The winning region of Player 0 in the safety game (G, W_0) is W_0,
   (b) If f_0 is a winning strategy for Player 0 in the safety game for (G, W_0), then f_0 is also a winning strategy for Player 0 in the Buchi game for (G, F),
   (c) the winning region of Player 1 in the guaranty game for (G, W_0) is W_1, and
   (d) if f_1 is a winning strategy for Player 1 in the guaranty game for (G, W_0), then f_1 is also a winning strategy for Player 1 in the Buchi game (G, F).