# Parity Games

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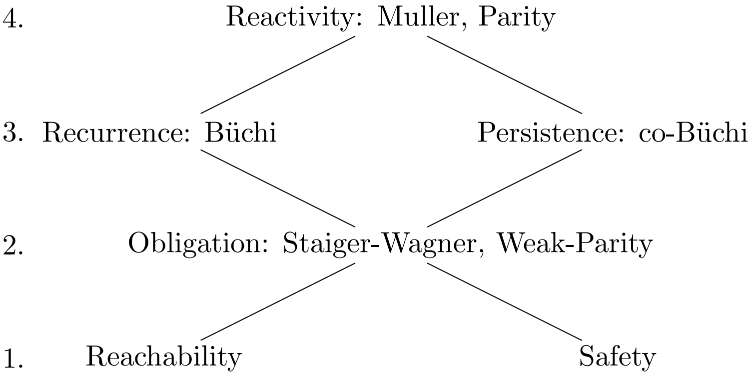
Grenoble, December 2018

# Hierarchy


4. Reactivity: Muller, Parity
3. Recurrence: Büchi
Persistence: co-Büchi
2. Obligation: Staiger-Wagner, Weak-Parity
1. Reachability
Safety

# Parity Games

A Parity game is a pair $(G, p)$, where

- $G = (S, S_0, E)$ is a game graph and
- $p : S \to \{0, \ldots, k\}$ is a priority function mapping every state in $S$ to a number in $\{0, \ldots, k\}$.

A play $\rho$ is winning for Player 0 iff the minimum priority visited infinitely often in $\rho$ is even: $\min_{s \in \mathrm{Inf}(\rho)} p(s)$ is even.

# Parity Games

## Theorem

1. *Parity games are determined (i.e., each state belongs to* $W_0$ *or* $W_1$*), and the has a positional winning strategy.*
2. *Over finite graphs, the winning regions and winning strategies of the two players can be effectively computed.*

# Overview

We will show two proofs:

- One for general (even infinite) game graph
- One constructive for finite game graphs to establish effectiveness.

# Proof 1

Given $G = (S, S_0, E)$ with priority function $p : S \to \{0, \ldots, d\}$ and let $P_i = \{s \in S \mid p(s) = i\}$. We proceed by induction on the number of priorities.

- Basis case: we either have an even or an odd priority

# Proof 1

Given $G = (S, S_0, E)$ with priority function $p : S \to \{0, \dots, d\}$ and let $P_i = \{s \in S \mid p(s) = i\}$. We proceed by induction on the number of priorities

- Basis case: we either have an even or an odd priority
- Induction step: we assume that the minimum priority $k$ is even (otherwise switch the roles of players 0 and 1 below).
  Let $\Pi_1$ be the set of vertices from which player 1 has a positional winning strategy.
  Show that from each vertex in $S \setminus \Pi_1$, player 0 has a positional winning strategy.

# Proof 1: Induction step

Consider the subgame with vertex set $S \setminus \Pi_1$

- Case 1: $S \setminus \Pi_1$ does not reach the minimal priority $k$.

  Then, $S \setminus \Pi_1$ defines a subgame. Why?

  Induction hypothesis applies.

- Case 2: $S \setminus \Pi_1$ contains vertices of minimal (even) priority.

  Then, $S \setminus (\Pi_1 \cup \mathrm{Attr}_0(P_k \setminus \Pi_1))$ defines a subgame

# Proof 1: Induction step

Player 0 can guarantee that starting from a vertex in $S \setminus \Pi_1$ the play remains there.

Either the play stays in $S \setminus (\Pi_1 \cup \mathrm{Attr}_0(P_k \setminus \Pi_1))$ from some point on, or it visits $\mathrm{Attr}_0(P_k \setminus \Pi_1)$ infinitely often.

In the first case player 0 wins by induction hypothesis with a positional strategy, in the second case by infinitely many visits to the lowest (even) priority, also with a positional strategy.

Altogether: Player 0 wins from each vertex in $S \setminus \Pi_1$ with a positional strategy.

# Proof 2

Given $G = (S, S_0, E)$ with $S$ finite and priority function $p : S \to \{0, \dots, d\}$. We proceed by induction on the number of states denoted by $n$.

- Basis case: we either have one Player-0 or Player-1 state with a selfloop (Note that every state in a game has at least one outgoing edge). Then the priority of the state determines if $S = W_0$ or $S = W_1$.
- Induction step: Let $P_i = \{s \mid p(s) = i\}$ be the set of states with priority $i$. Assume $P_0 \neq \emptyset$, otherwise assume $P_1 \neq \emptyset$ and switch the roles of Players 0 and 1 below. Finally, if $P_0 = P_1 = \emptyset$ decrease every priority by 2.

# Proof (induction step cont.)

Choose $s \in P_0$ and let $X = \text{Attr}_0(\{s\})$. Note that $S \setminus X$ is a subgame with $< n$ states.

The induction hypothesis gives a partition of $S \setminus X$ into winning regions $U_0$ and $U_1$ for Player 0 and 1, respectively, and corresponding positional winning strategies.

- Case 1: Player 0 can guarantee a transition from $s$ to $U_0 \cup X$, i.e., if $s \in S_0$, then there exists $s' \in U_0 \cup X$ such that $(s, s') \in E$ or if $s \in S_1$, then for all $(s, s') \in E$, $s' \in U_0 \cup X$ holds.

  Claim:

  (i) $U_0 \cup X \subseteq W_0$

  (ii) $U_1 \subseteq W_1$.

# Proof (Case 1 cont.)

The positional strategy for Player 0 on $U_0 \cup X$ is:

1. On $U_0$ play according to the positional strategy given by the induction hypothesis
2. On $X$ $(= \mathrm{Attr}_0(\{s\}))$ play according to the attractor strategy. Then eventually reach $s$
3. From $s$ "move back" to $U_0 \cup X$.

For Player 1 use the positional strategy on $U_1$ given by the induction hypothesis.

Proof of claim: (ii) is clear, since starting in $U_1$ Player 1 can guarantee that the play remains in $U_1$ (see picture). For (i), the play remains in $U_0 \cup X$ if the strategy for state $s$ is followed. If the play eventually remains in $U_0$, then Player 0 wins by induction hypothesis, otherwise the play passes through $s$ infinitely often, which is winning as well.

# Proof (Case 2)

- Case 2: Player 1 can guarantee a transition to $U_1$ from $s$, i.e., if $s \in S_0$, then all edges $(s, s') \in E$ lead to $U_1$ $(s' \in U_1)$, and if $s \in S_1$, then there exists $s' \in U_1$ such that $(s, s') \in E$.

  Let $Y = \mathrm{Attr}_1(U_1)$, then $s \in Y$ and $S \setminus Y$ is a subgame with $< n$ states. The induction hypothesis gives winning region $V_0$ and $V_1$ and corresponding positional winning strategies.

  Claim:

  (i) $V_0 \subseteq W_0$

  (ii) $V_1 \cup Y \subseteq W_1$.

  Proof of claim: (i) is clear, since Player 0 can guarantee to stay within $V_0$. For (ii), for all states in $Y$, Player 1 can guarantee to move to $U_1$ and remind there. From $t \in V_1$ Player 0 can either move to $Y$ or stay in $V_1$. Both choices are winning for Player 1.

# Example